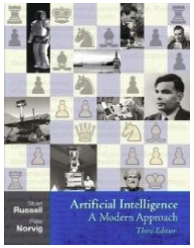
Outline

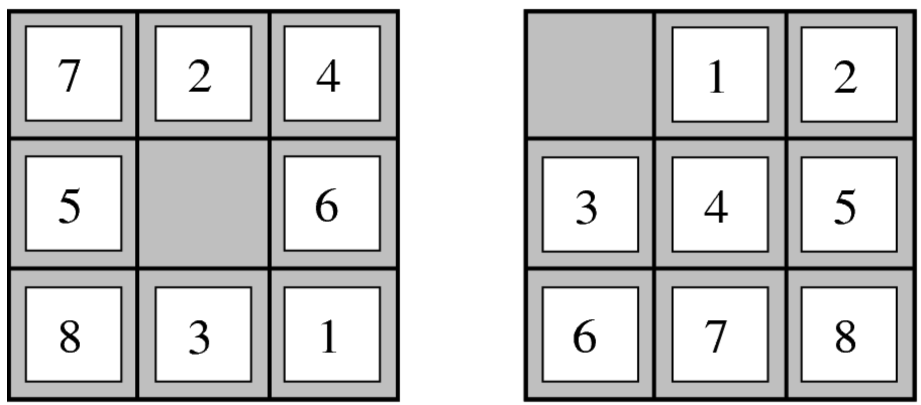
- Best-first search
  - Greedy best-first search
  - A* search
  - Heuristics
  - Admissible Heuristics
  - Graph Search
  - Consistent Heuristics
- Local search algorithms
  - Hill-climbing search
  - Beam search
  - Simulated annealing search
  - Genetic algorithms
- Constraint Satisfaction Problems



Many slides based on Russell & Norvig's slides *Artificial Intelligence: A Modern Approach*

Motivation

- Uninformed search algorithms are too inefficient
  - they expand far too many unpromising paths
- Example:
  - 8-puzzle



Start State Goal State

  - Average solution depth = 22
  - Breadth-first search to depth 22 has to expand about 3.1×10^{10} nodes

→ try to be more clever with what nodes to expand

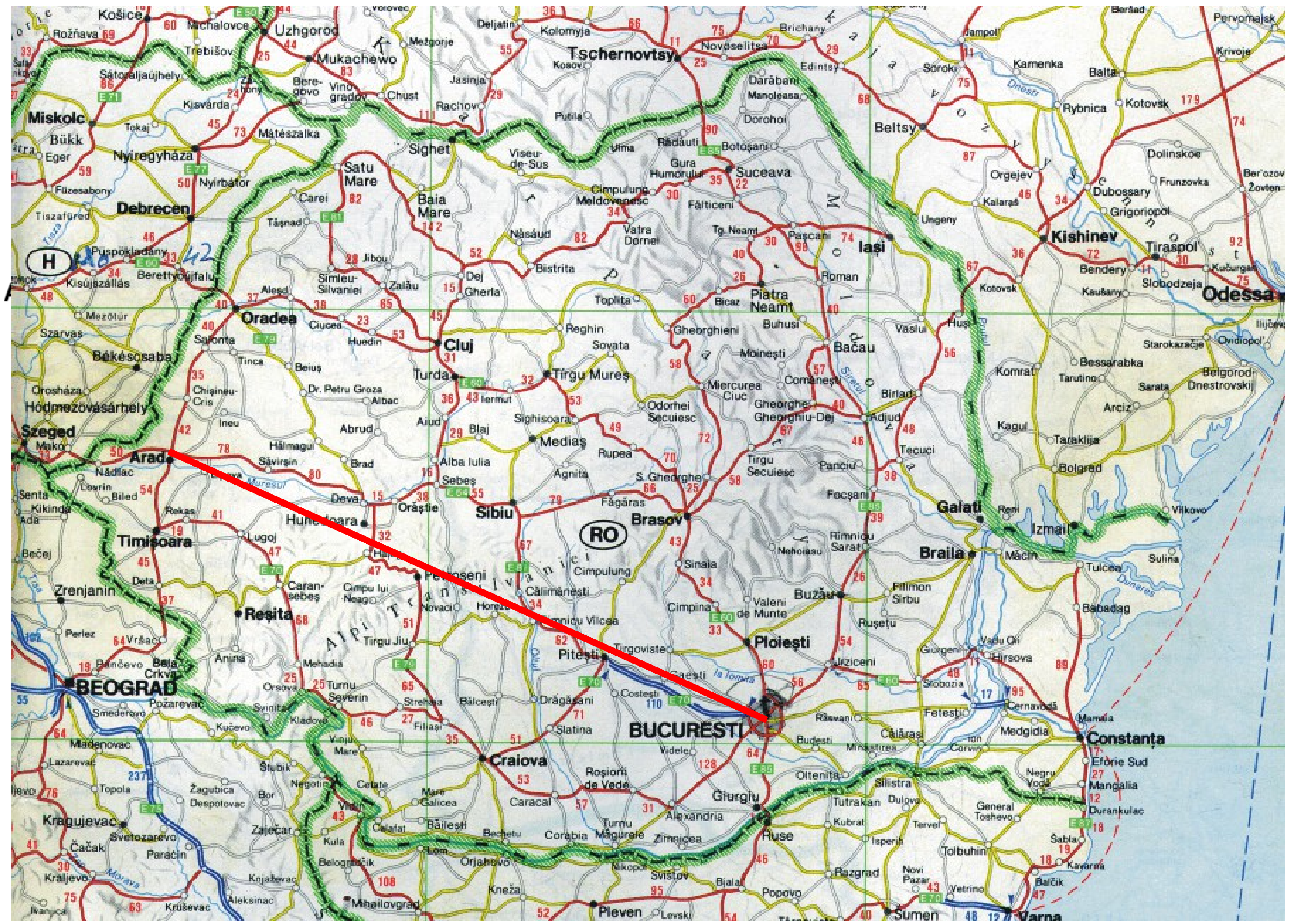
Best-First Search

- Recall
  - Search strategies are characterized by the order in which they expand the nodes of the search tree
  - Uninformed tree-search algorithms sort the nodes by problem-independent methods (e.g., recency)
- Basic Idea of Best-First Search
  - use a heuristic evaluation function $f(n)$ for each node
    - estimate of the "desirability" of the node's state
  - expand most desirable unexpanded node
- Implementation
  - use Tree Search algorithm
  - order the nodes in fringe in decreasing order of desirability
- Algorithms
  - Greedy best-first search
  - A* search

Heuristic

- Greek "heurisko" (εὑρίσκω) → "I find"
  - cf. also „Eureka!“
- informally denotes a „rule of thumb“
  - i.e., knowledge that may be helpful in solving a problem
  - note that heuristics may also go wrong!
- In tree-search algorithms, a heuristic denotes a function that estimates the remaining costs until the goal is reached
- Example:
  - straight-line distances may be a good approximation for the true distances on a map of Romania
  - and are easy to obtain (ruler on the map)
    - but cannot be obtained directly from the distances on the map

Romania Example: Straight-line Distances



Straight−line distance to Bucharest	
Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	176
Giurgiu	77
Hirsova	151
Iasi	226
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Mehadia	241
Neamt	234
Oradea	380
Pitesti	98
Rimnicu Vilcea	193
Sibiu	253
Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

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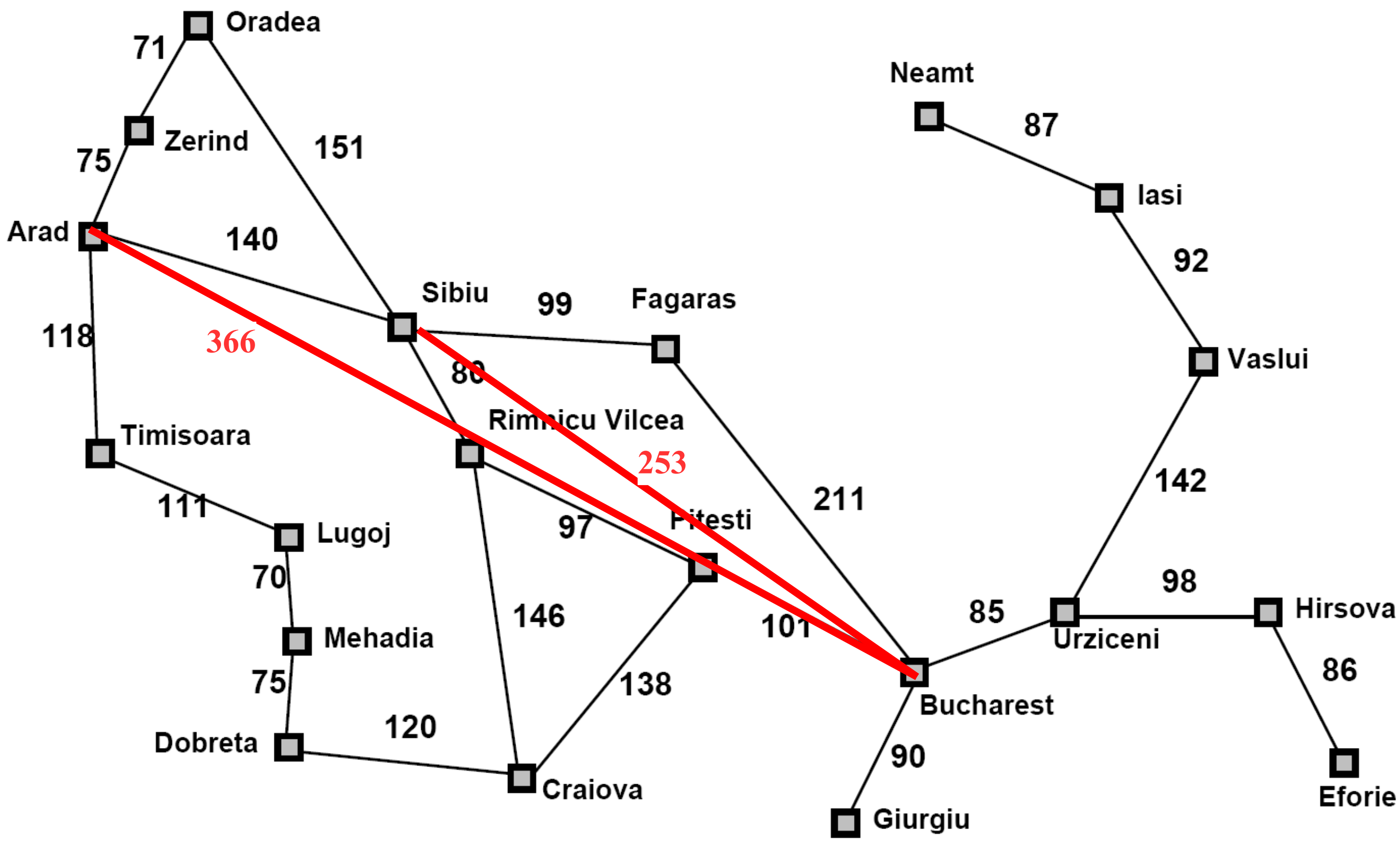

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Greedy Best-First Search

- Evaluation function $f(n) = h(n)$ ($h$euristic)
  - estimates the cost from node n to $goal$
  - e.g., $h_{SLD}(n)$ = straight-line distance from n to Bucharest
- Greedy best-first search expands the node that appears to be closest to goal
  - according to evaluation function
- Example:

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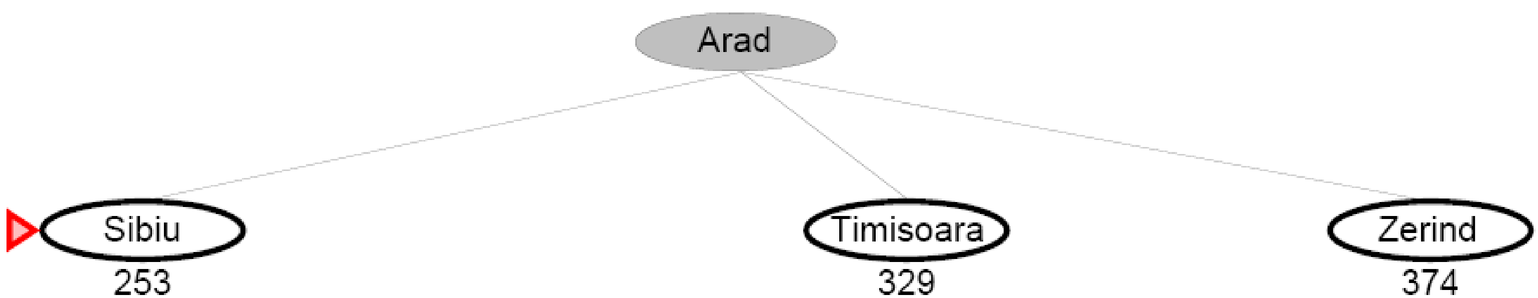

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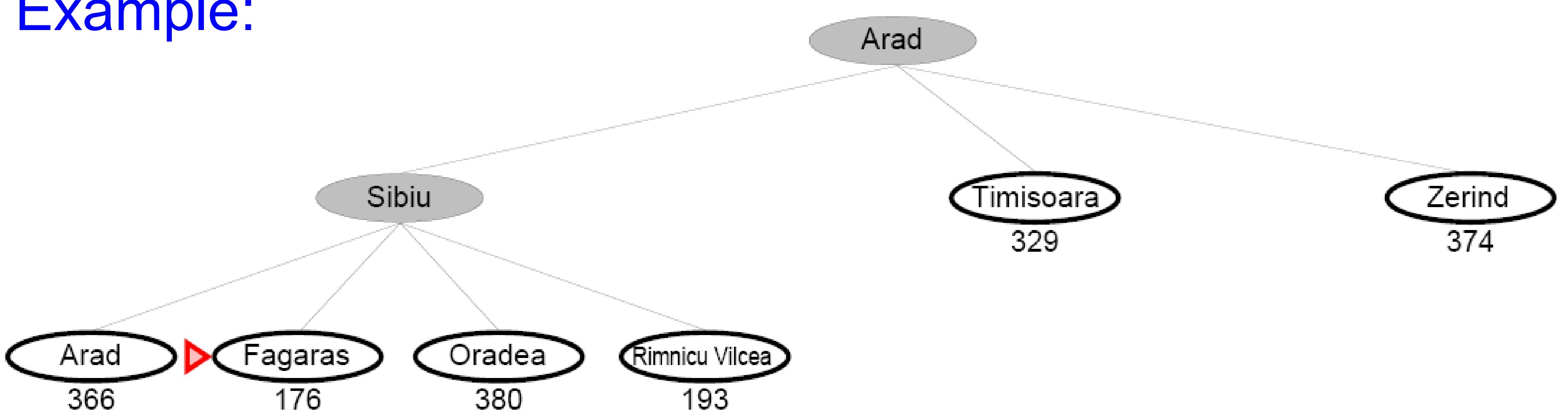

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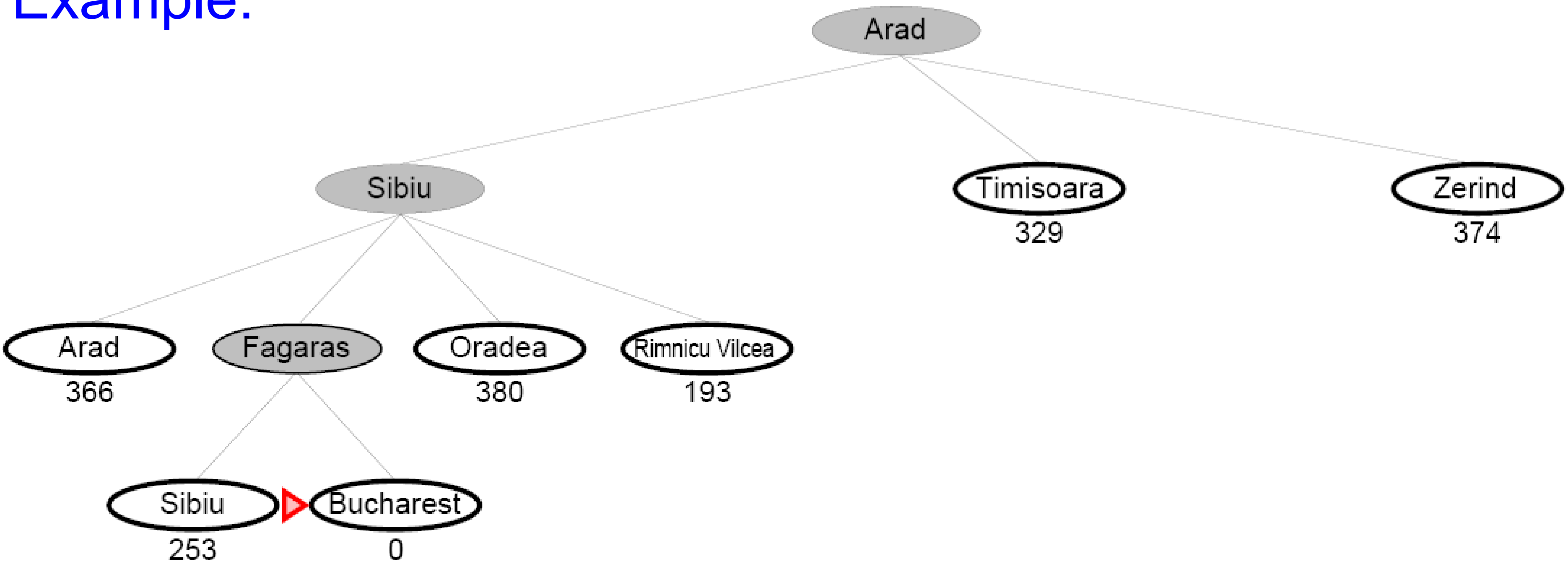
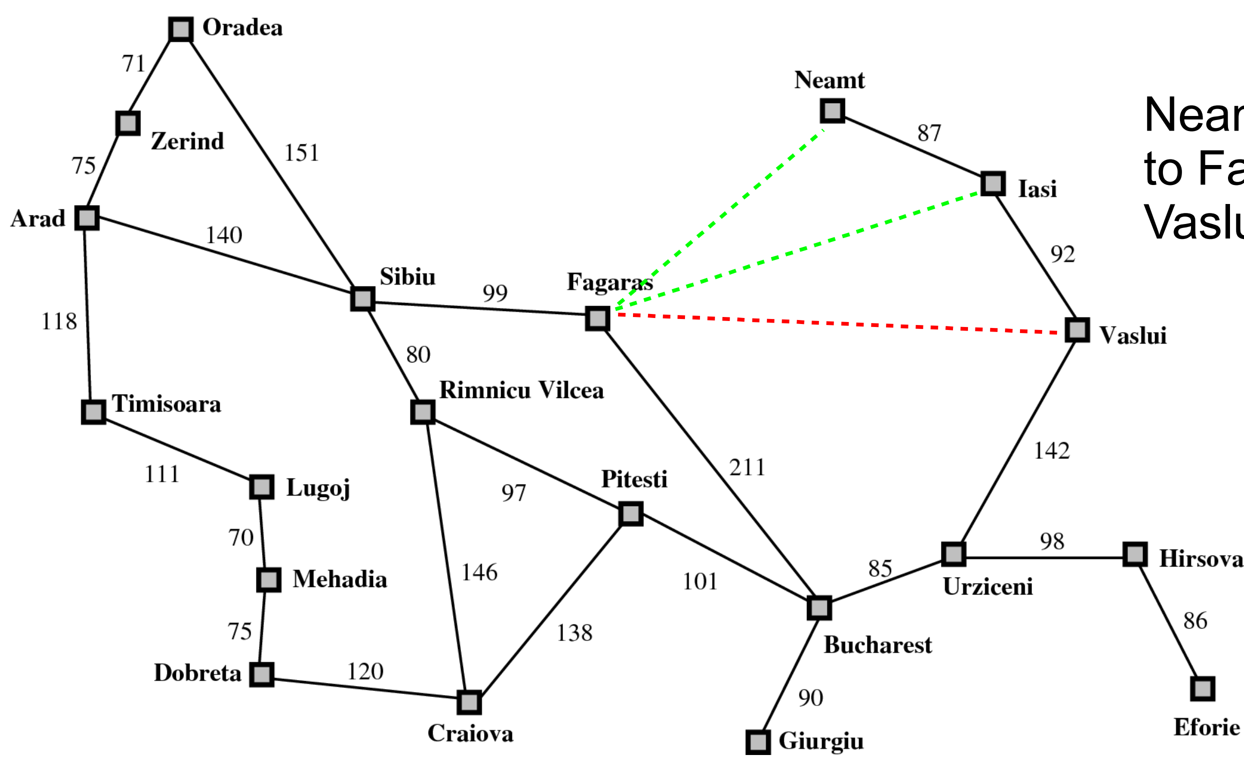


Properties of Greedy Best-First Search

- Completeness
  - No – can get stuck in loops
  - Example: We want to get from Iasi to Fagaras
    - Iasi → Neamt → Iasi → Neamt → ...

Note:
These two are different search nodes referring to the same state!



Neamt is closer to Fagaras than Vaslui

Properties of Greedy Best-First Search

- Completeness
  - No – can get stuck in loops
  - can be fixed with careful checking for duplicate states

  → complete in finite state space with repeated-state checking
- Time Complexity
  - $O(b^m)$, like depth-first search
  - but a good heuristic can give dramatic improvement
    - optimal case: best choice in each step → only d steps
    - a good heuristic improves chances for encountering optimal case
- Space Complexity
  - has to keep all nodes in memory → same as time complexity
- Optimality
  - No
  - Example:
    - solution Arad → Sibiu → Fagaras → Bucharest is not optimal

A* Search

- Best-known form of best-first search

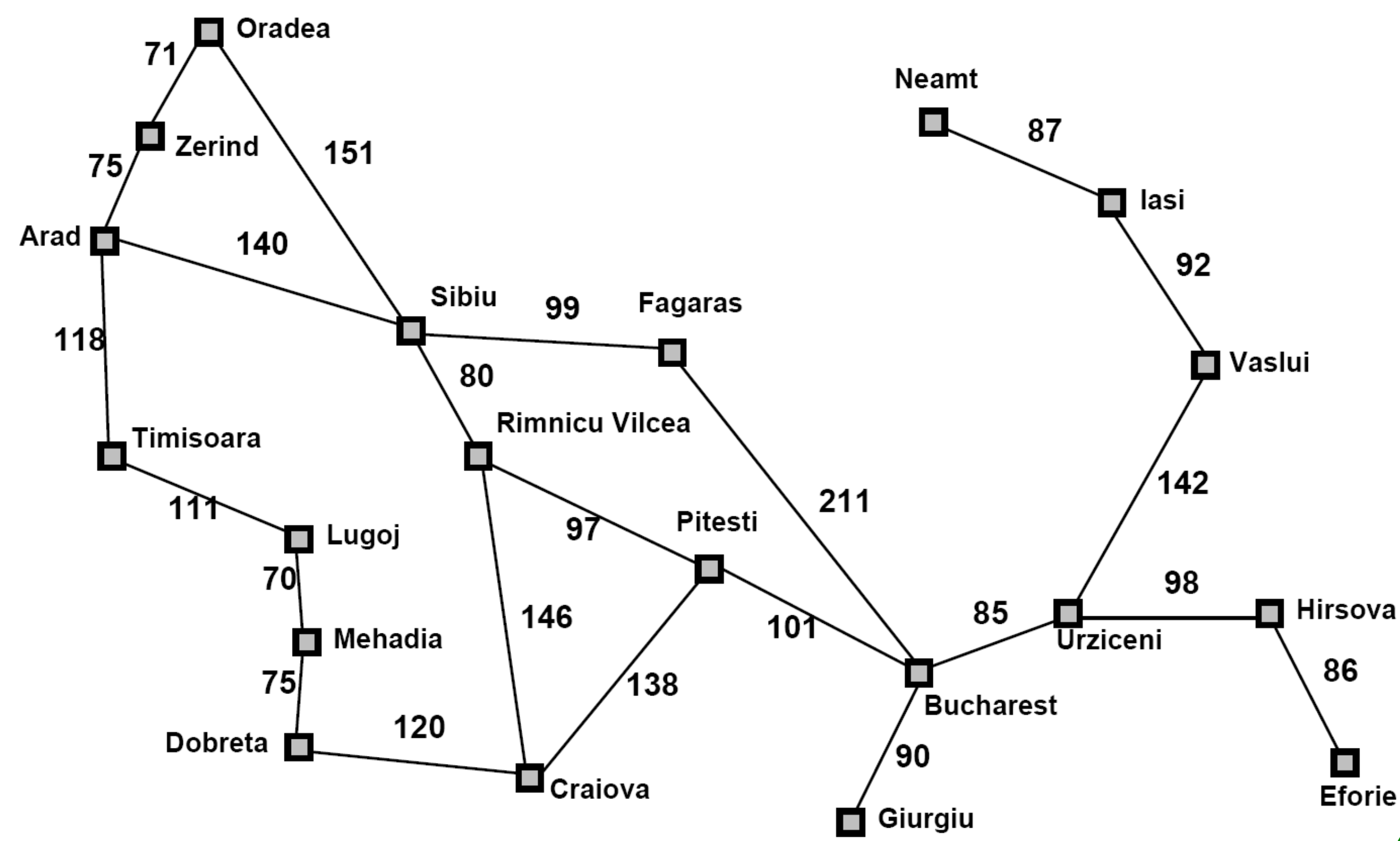
- Basic idea:
  - avoid expanding paths that are already expensive

  $\rightarrow$ evaluate complete path cost not only remaining costs

- Evaluation function: $f(n) = g(n) + h(n)$
  - $g(n)$ = cost so far to reach node n
  - $h(n)$ = estimated cost to get from n to goal
  - $f(n)$ = estimated cost of path to goal via n

Beispiel

$g(n)$



$h(n)$

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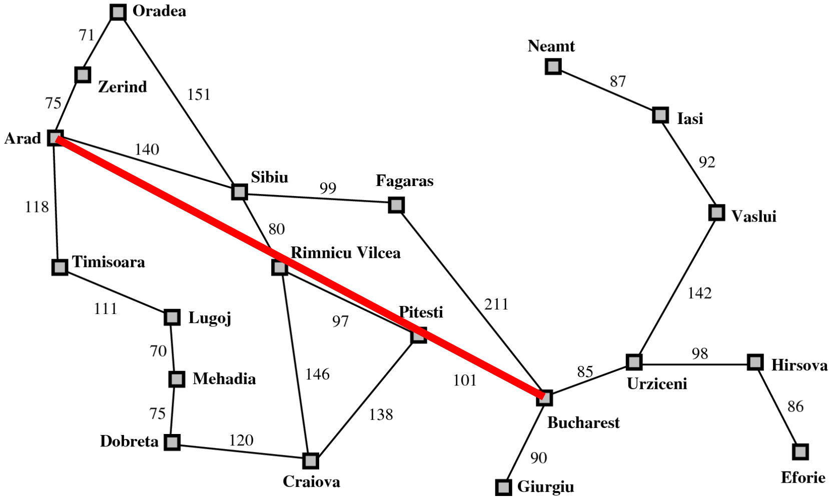
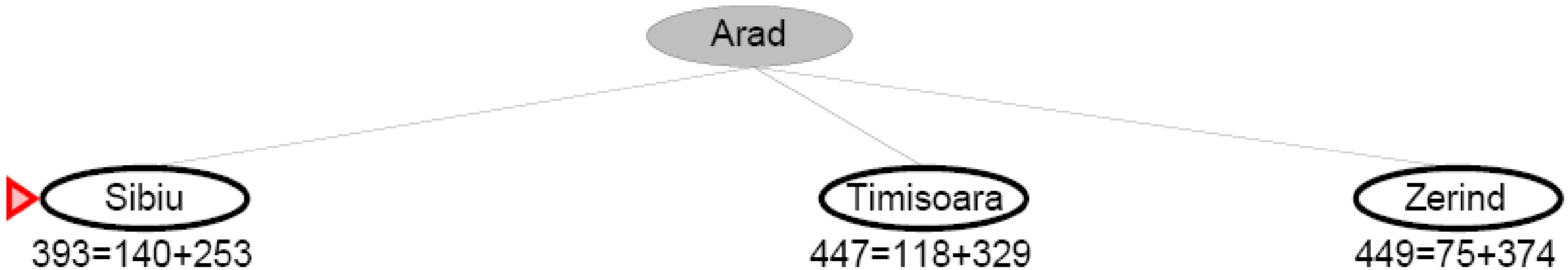
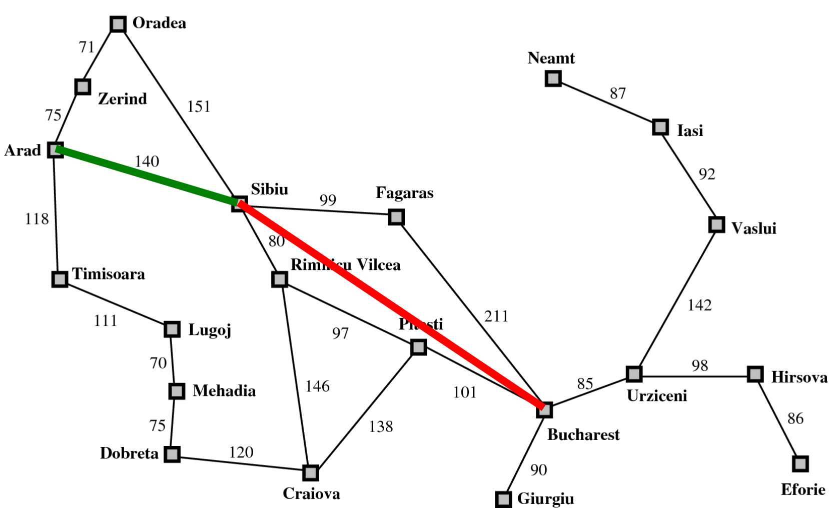
A* Search Example

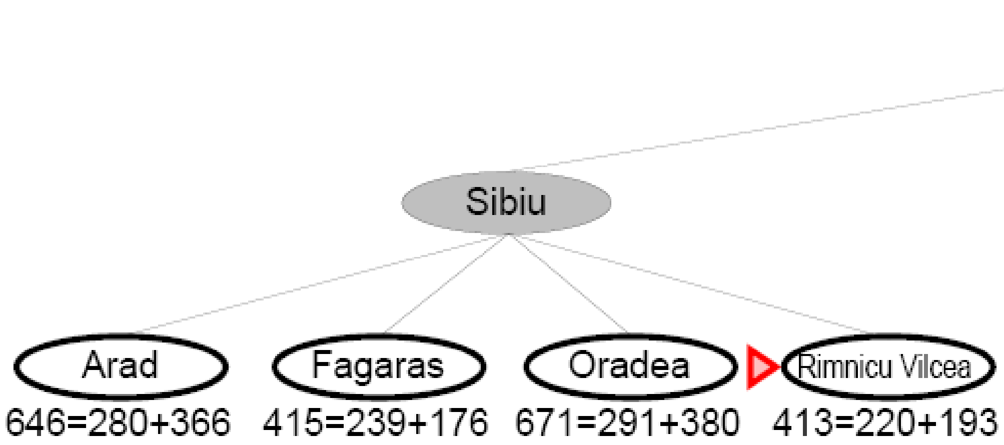


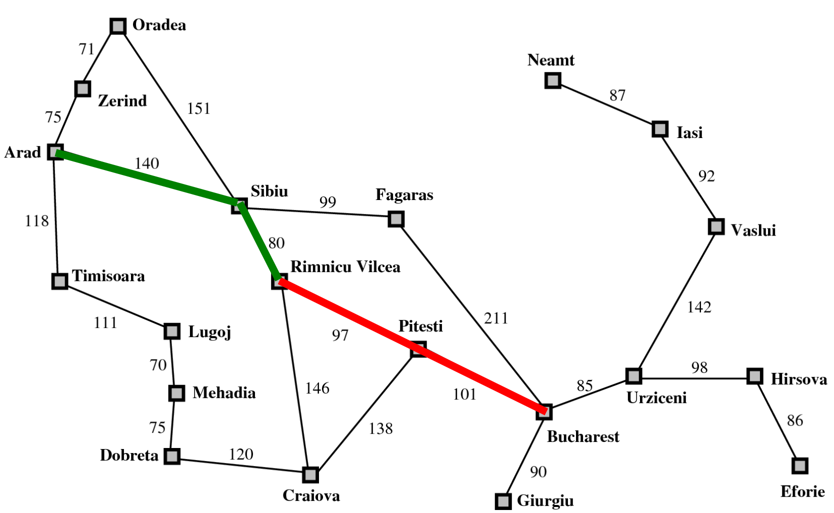
A* Search Example





A* Search Example





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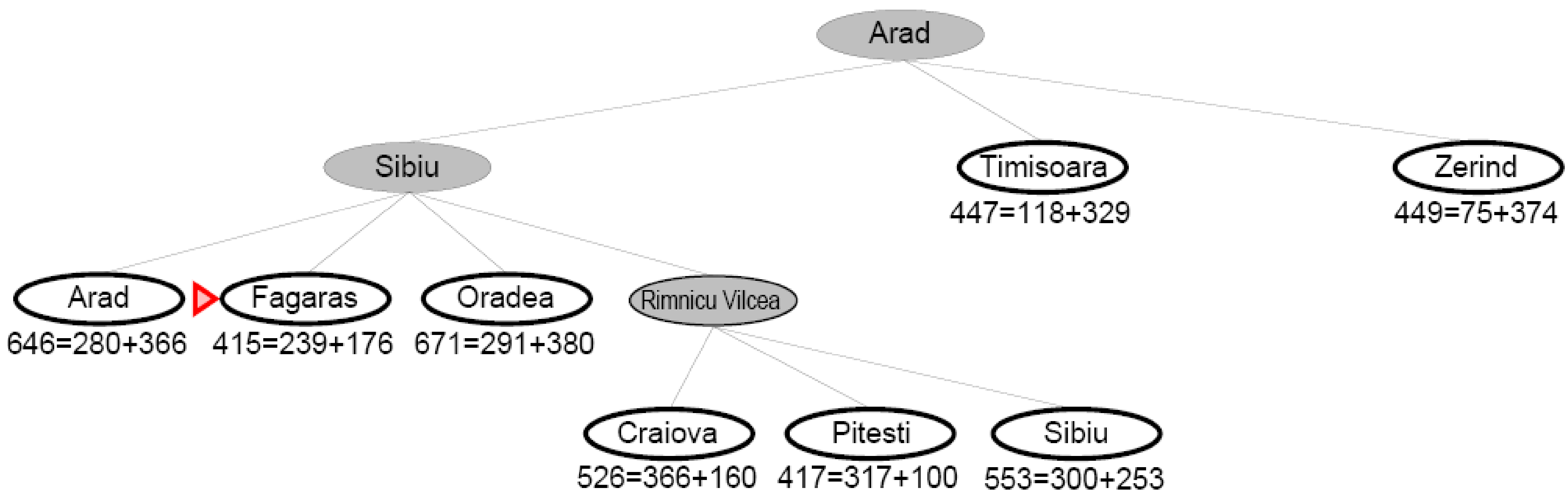
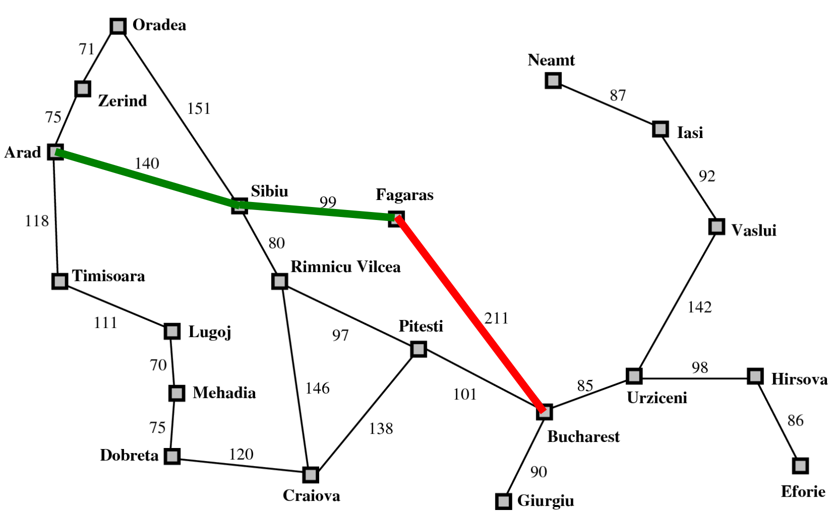
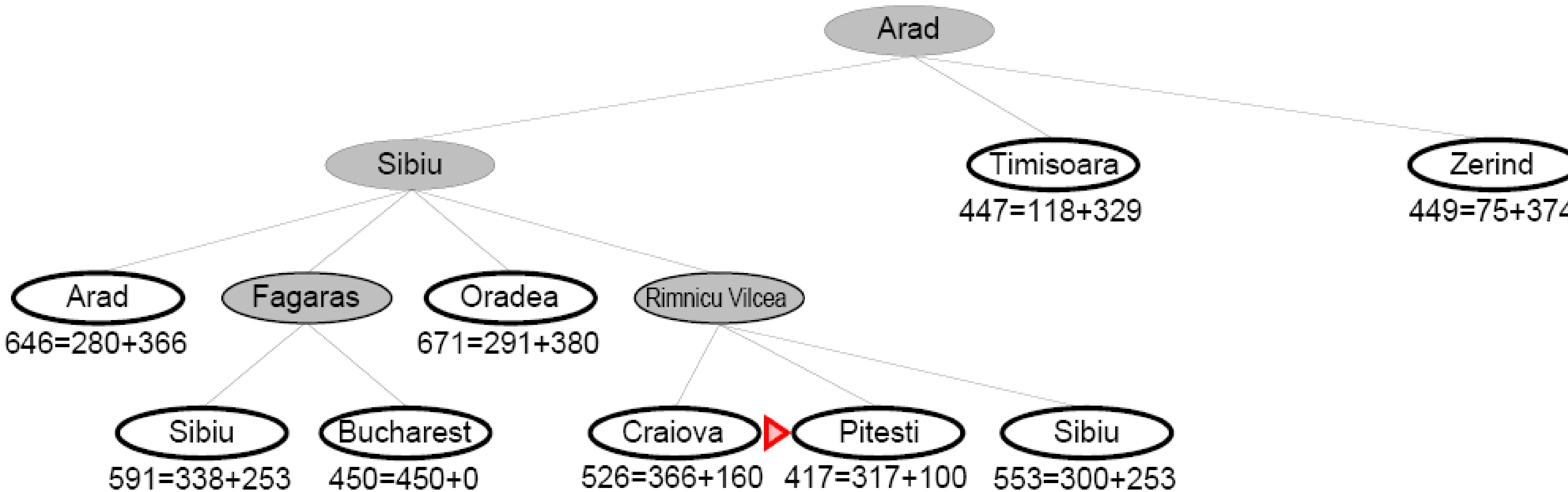
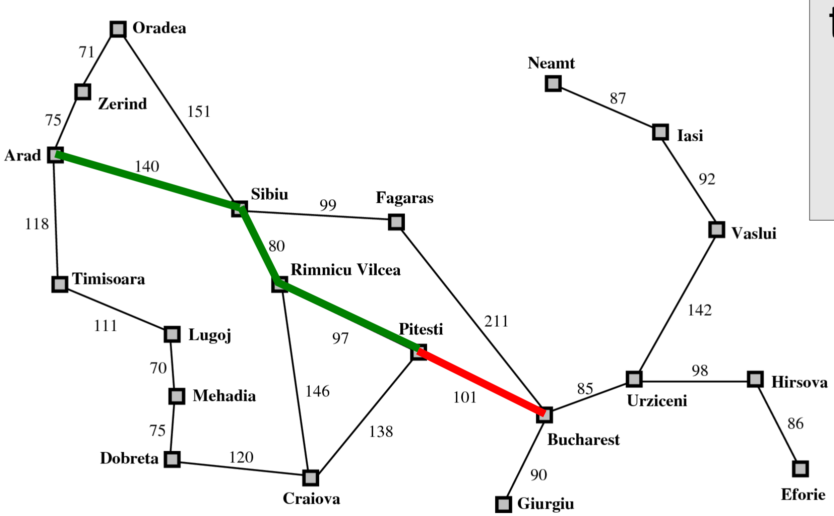




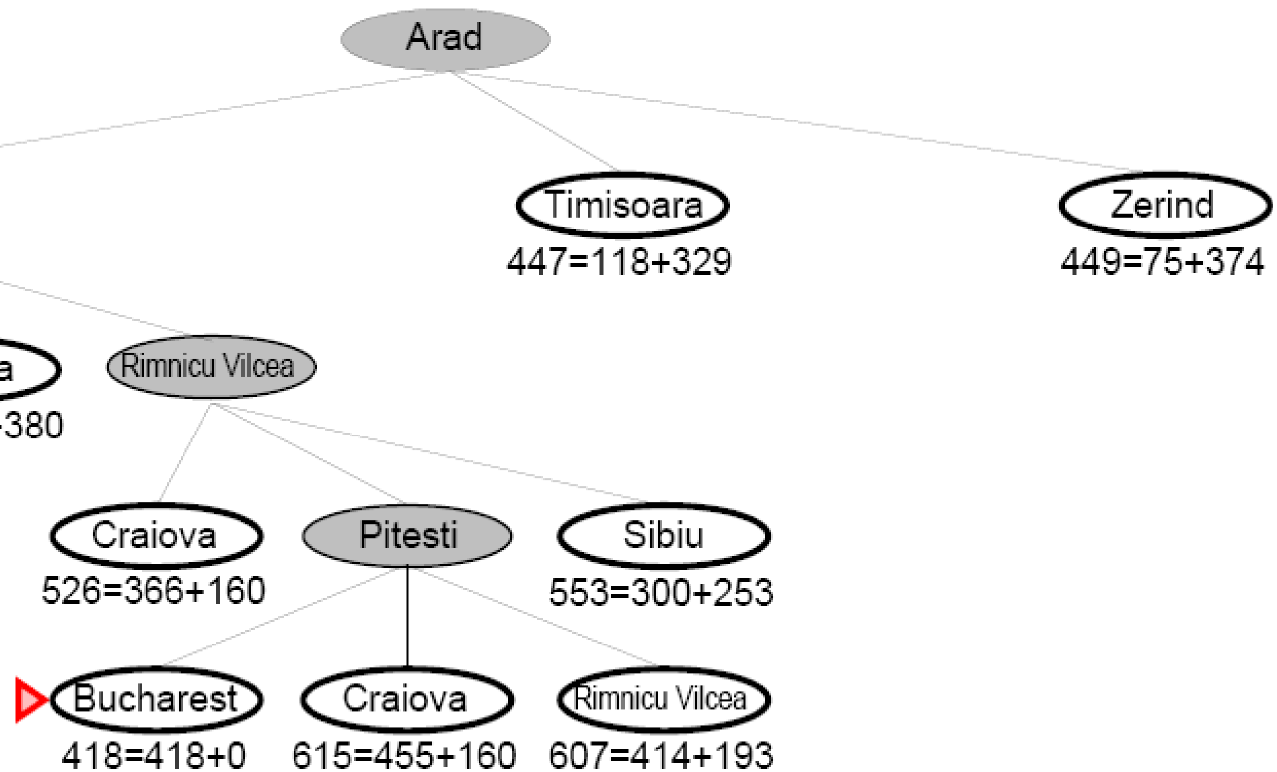
A* Search Example

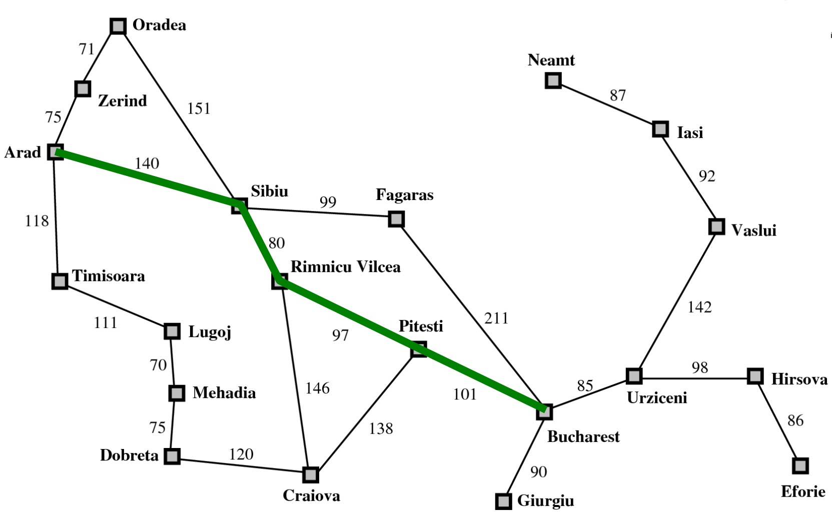




Note that Pitesti will be expanded even though Bucharest is already in the fringe! This is good, because we may still find a shorter way to Budapest. Greedy Search would not do that.

A* Search Example





Properties of A*

- Completeness
  - Yes
  - unless there are infinitely many nodes with $f(n) \leq f(G)$
- Time Complexity
  - it can be shown that the number of nodes grows exponentially unless the error of the heuristic $h(n)$ is bounded by the logarithm of the value of the actual path cost $h^*(n)$, i.e.

$$|h(n) - h^*(n)| \leq O(\log h^*(n))$$

- Space Complexity
  - keeps all nodes in memory
  - typically the main problem with A*
- Optimality
  - ???

→ following pages

Admissible Heuristics

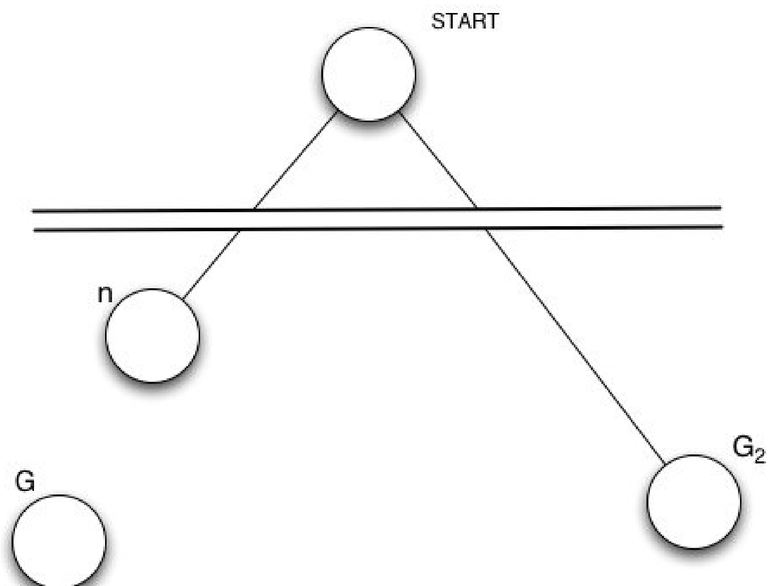
A heuristic is admissible if it *never* overestimates the cost to reach the goal

- Formally:
  - $h(n) \leq h^*(n)$ if $h^*(n)$ are the true cost from n to goal
- Example:
  - Straight-Line Distances h_{SLD} are an admissible heuristics for actual road distances h^*
- Note:
  - $h(n) \geq 0$ must also hold, so that $h(goal) = 0$

Theorem

If $h(n)$ is admissible, A* using `TREE-SEARCH` is optimal.

Proof:



Let n be an unexpanded node in the fringe such that n is on a shortest path to an optimal goal G with path cost $C^* = g(G)$.

Suppose some suboptimal goal G_2 has been generated and is in the fringe.

$C^* = g(n) + h^*(n)$

$f(n) = g(n) + h(n)$

$h(n) \leq h^*(n)$

because h admissible

$f(n) \leq C^* < f(G_2)$

$g(G_2) > C^*$

because G_2 suboptimal

$f(G_2) = g(G_2)$

because $h(G_2) = 0$

(holds for all goal nodes)

G_2 will never be expanded, and G will be returned

A* and Graph Search

- So far we only had tree search where each node in the search tree represents one possible path to the associated domain state
  - e.g., one can go directly from Arad to Sibiu or via Oradea
- Problem:
  - In some cases the path that is detected later may be the better path
  - so that a previously found solution starting from Sibiu has to re-investigated with the new, cheaper path
- Two solutions
  - Add ability to detect repeated states
    - → graph search
    - general solution that also works for BFS, DFS, ...
  - Or ensure that the cheaper path is always taken first
    - → consistent heuristics
    - specific solution for A*

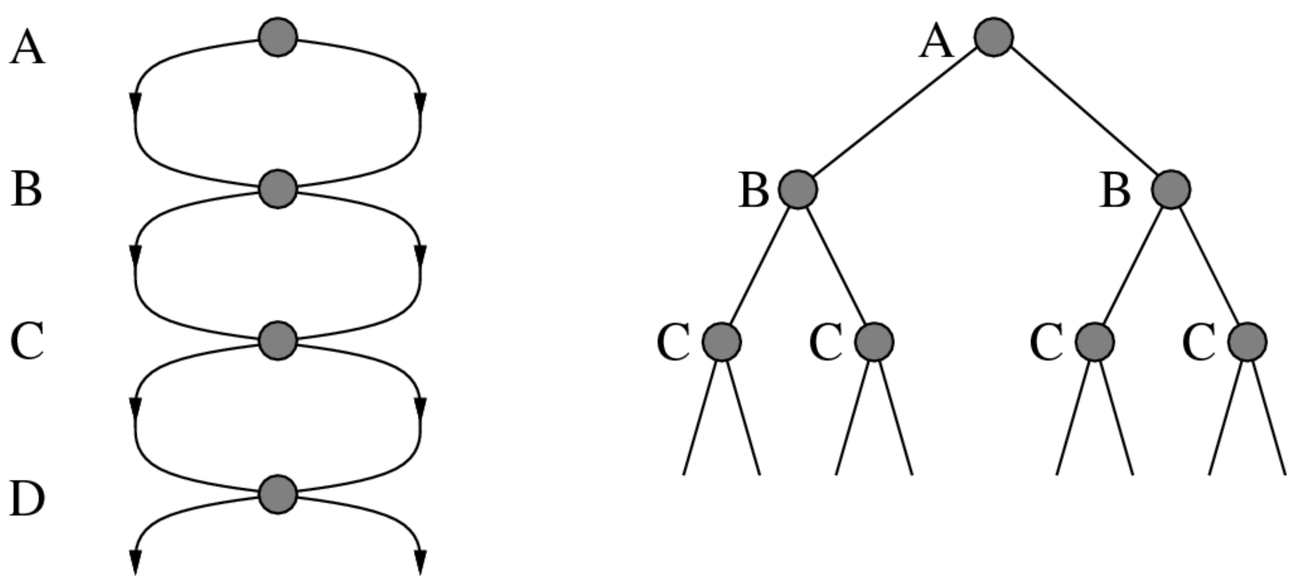
Repeated States

- Failure to detect repeated states can turn a linear problem into an exponential one!

Ribbon Example

- two connections from each state to the next

d states but state space is 2^d



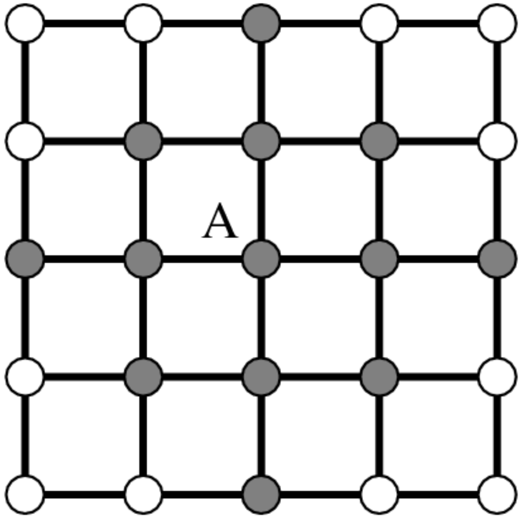
(a) (b)

Repeated States

- Failure to detect repeated states can turn a linear problem into an exponential one!

(more realistic) Grid Example

- each square on grid has 4 neighboring states in
- thus, game tree w/o repetitions has 4^d nodes
- but only about $2d^2$ different states are reachable in d steps



Graph Search

- remembers the states that have been visited in a list *closed*
  - Note: the fringe list is often also called the open list

```
function GRAPH-SEARCH( problem, fringe) returns a solution, or failure
   closed ← an empty set
   fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
   loop do
       if fringe is empty then return failure
       node ← REMOVE-FRONT(fringe)
       if GOAL-TEST(problem, STATE[node]) then return node
       if STATE[node] is not in closed then
           add STATE[node] to closed
           fringe ← INSERTALL(EXPAND(node, problem), fringe)
   end
```

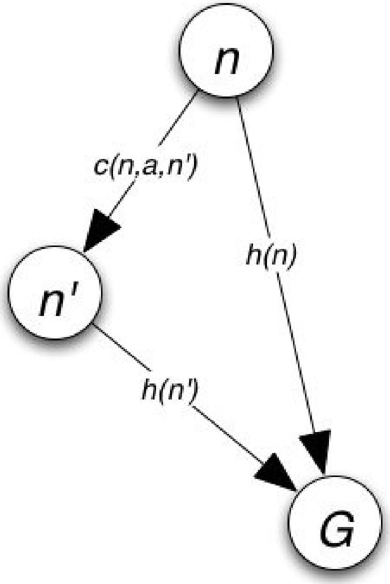
- Example:
  - Dijkstra's algorithm is the graph-search variant of uniform cost search

Consistent Heuristics

- Graph-Search discards new paths to repeated state even though the new path may be cheaper
  → Previous proof breaks down
- 2 Solutions
  1. Add extra bookkeeping to remove the more expensive path
  2. Ensure that optimal path to any repeated state is always followed first
- Requirement for Solution 2:

A heuristic is consistent if for every node n and every successor n' generated by any action a it holds that

$$h(n) \leq c(n, a, n') + h(n')$$



Lemma 1

Every consistent heuristic is admissible.

Proof Sketch by induction

Base Case: for all nodes n, in which an action a leads to goal G

$$h(n) \leq c(n, a, G) + h(G) = h^*(n)$$

By *induction on the path length from goal*, this argument can be extended to all nodes, so that eventually

$$\forall n : h(n) \leq h^*(n)$$

- Note:
  - not every admissible heuristic is consistent
  - but most of them are
    - it is hard to find non-consistent admissible heuristics

Lemma 2

If $h(n)$ is consistent, then the values of $f(n)$ along any path are non-decreasing.

Proof:

$$f(n) = g(n) + h(n) \leq g(n) + c(n, a, n') + h(n') =$$

$$g(n) + c(n, a, n') + h(n') = g(n') + h(n') = f(n')$$

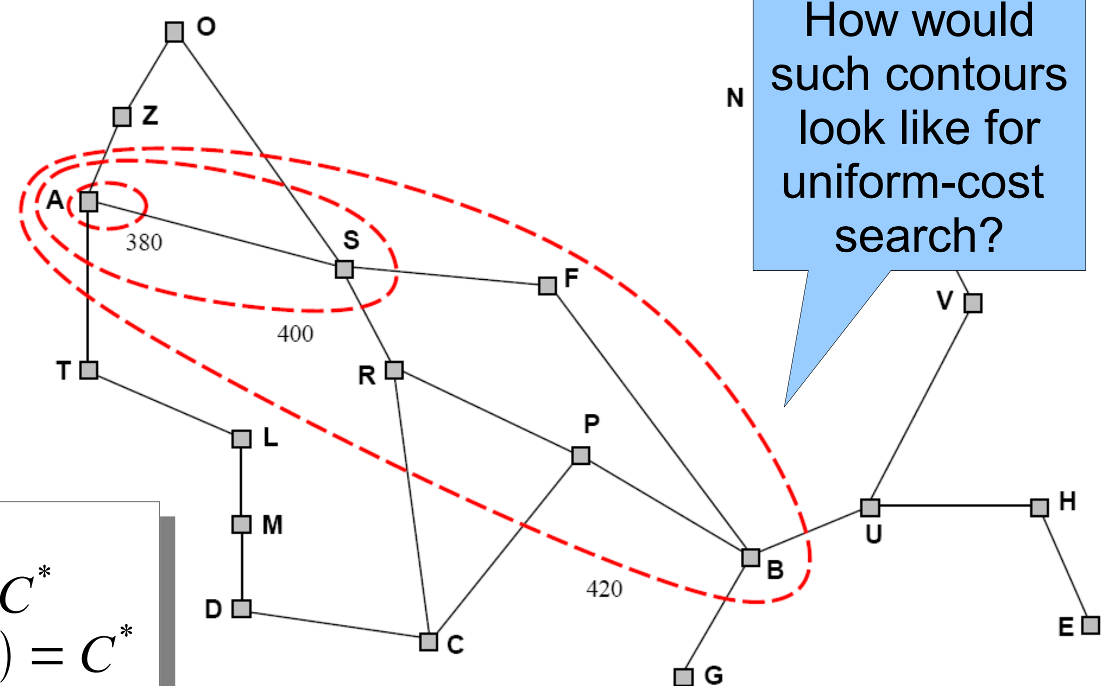
Theorem

If $h(n)$ is consistent, A* is optimal.

Proof:

A* expands nodes in order of increasing f value

Contour labelled f_i contains all nodes with $f(n) < f_i$



How would such contours look like for uniform-cost search?

Contours expand gradually
Cannot expand f_{i+1} until f_i is finished.

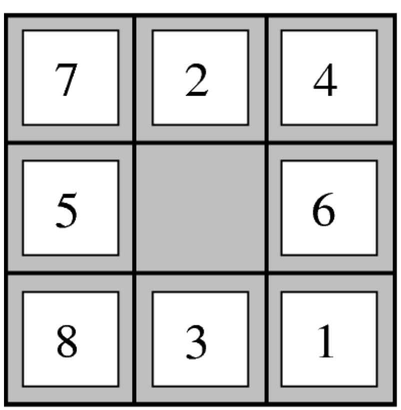
Eventually
- A* expands all nodes with $f(n) < C^*$
- A* expands some nodes with $f(n) = C^*$
- A* expands no nodes with $f(n) > C^*$

Memory-Bounded Heuristic Search

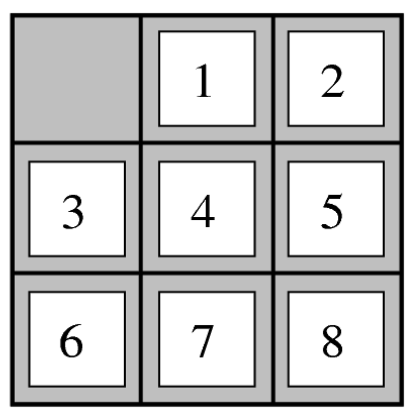
- Some solutions to A* space problems
  (maintaining completeness and optimality)
  - Iterative-deepening A* (IDA*)
    - like iterative deepening
    - cutoff information is the f-cost $(g + h)$ instead of depth
  - Recursive best-first search (RBFS)
    - recursive algorithm that attempts to mimic standard best-first search with linear space.
    - keeps track of the f-value of the best alternative path available from any ancestor of current node
    - heuristic evaluations are updated with results of successors
  - (Simple) Memory-bounded A* ((S)MA*)
    - drop the worst leaf node when memory is full
    - its value will be updated to its parent
    - May need to be re-searched later

Admissible Heuristics: 8-Puzzle

- $h_{\mathrm{MIS}}(n)$ = number of misplaced tiles
  - admissible because each misplaced tile must be moved at least once
- $h_{\mathrm{MAN}}(n)$ = total Manhattan distance
  - i.e., no. of squares from desired location of each tile
  - admissible because this is the minimum distance of each tile to its target square
- Example:



Start State



Goal State

$h_{MIS}(start) = 8$

$h_{MAN}(start) = 18$

$h^*(start) = 26$

Effective Branching Factor

- Evaluation Measure for a search algorithm:
  - assume we searched N nodes and found solution in depth d
  - the effective branching factor b^* is the branching factor of a uniform tree of depth d with $N+1$ nodes, i.e.

$$1+N = 1+b^* + (b^*)^2 + \ldots + (b^*)^d$$

- Measure is fairly constant for different instances of sufficiently hard problems
  - Can thus provide a good guide to the heuristic's overall usefulness.
  - A good value of b^* is 1

Efficiency of A* Search

- Comparison of number of nodes searched by A* and Iterative Deepening Search (IDS)
  - average of 100 different 8-puzzles with different solutions
  - **Note:** heuristic $h_2 = h_{MAN}$ is always better than $h_1 = h_{MIS}$

	Suchkosten			Effektiver Verzweigungsfaktor		
d	IDS	$A^*(h_1)$	$A^*(h_2)$	IDS	$A^*(h_1)$	$A^*(h_2)$
2	10	6	6	2,45	1,79	1,79
4	112	13	12	2,87	1,48	1,45
6	680	20	18	2,73	1,34	1,30
8	6384	39	25	2,80	1,33	1,24
10	47127	93	39	2,79	1,38	1,22
12	3644035	227	73	2,78	1,42	1,24
14	–	539	113	–	1,44	1,23
16	–	1301	211	–	1,45	1,25
18	–	3056	363	–	1,46	1,26
20	–	7276	676	–	1,47	1,27
22	–	18094	1219	–	1,48	1,28
24	–	39135	1641	–	1,48	1,26

Dominance

If h_1 and h_2 are admissible, h_2 dominates h_1 if $\forall n : h_2(n) \geq h_1(n)$

- if h_2 dominates h_1 it will perform better because it will *always* be closer to the optimal heuristic h^*
- Example:
  - h_{MAN} dominates h_{MIS} because if a tile is misplaced, its Manhattan distance is ≥ 1

Theorem: (Combining admissible heuristics)

If h_1 and h_2 are two admissible heuristics than

$$h(n) = max(h_1(n), h_2(n))$$

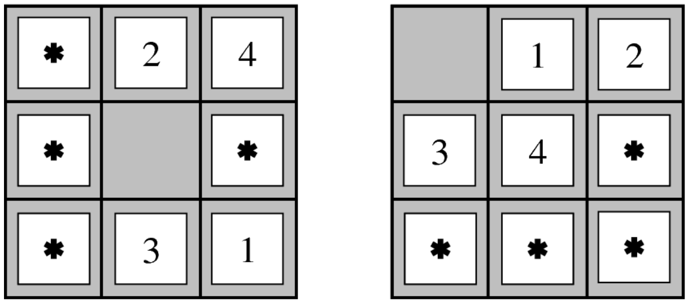
is also admissible and dominates h_1 and h_2

Relaxed Problems

- A problem with fewer restrictions on the actions is called a relaxed problem
- The cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem
- Examples:
  - If the rules of the 8-puzzle are relaxed so that a tile can move anywhere, then h_{MIS} gives the shortest solution
  - If the rules are relaxed so that a tile can move to any adjacent square, then h_{MAN} gives the shortest solution
- Thus, looking for relaxed problems is a good strategy for inventing admissible heuristics.

Pattern Databases

- Admissible heuristics can also be derived from the solution cost of a subproblem of a given problem.
  - This cost is a lower bound on the cost of the real problem.
- Pattern databases store the exact solution (length) for every possible subproblem instance
  - constructed once for all by searching backwards from the goal and recording every possible pattern
- Example:
  - store exact solution costs for solving 4 tiles of the 8-puzzle
  - sample pattern:



Start State Goal State

Learning of Heuristics

- Another way to find a heuristic is through learning from experience
- Experience:
  - states encountered when solving lots of 8-puzzles
  - states are encoded using features, so that similarities between states can be recognized
- Features:
  - for the 8-puzzle, features could, e.g. be
    - the number of misplaced tiles
    - number of pairs of adjacent tiles that are also adjacent in goal
    - ...
- An inductive learning algorithm can then be used to predict costs for other states that arise during search.
- No guarantee that the learned function is admissible!

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Summary

- Heuristic functions estimate the costs of shortest paths
- Good heuristics can dramatically reduce search costs
- Greedy best-first search expands node with lowest estimated remaining cost
  - incomplete and not always optimal
- A* search minimizes the path costs so far plus the estimated remaining cost
  - complete and optimal, also optimally efficient:
    - no other search algorithm can be more efficient, because they all have search the nodes with $f(n) < C^*$
    - otherwise it could miss a solution
- Admissible search heuristics can be derived from exact solutions of reduced problems
  - problems with less constraints
  - subproblems of the original problem