Outline

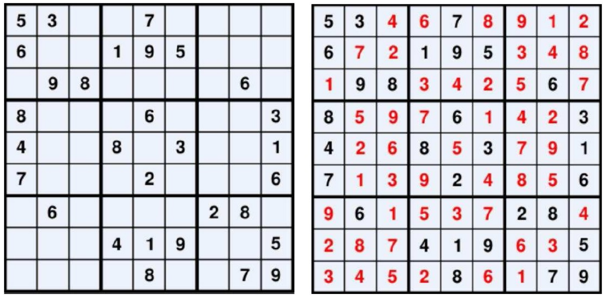
- Best-first search
  - Greedy best-first search
  - A* search
  - Heuristics
- Local search algorithms
  - Hill-climbing search
  - Beam search
  - Simulated annealing search
  - Genetic algorithms
- Constraint Satisfaction Problems
  - Constraints
  - Constraint Propagation
  - Backtracking Search
  - Local Search

Constraint Satisfaction Problems

Special Type of search problem:

- state is defined by variables X_i with values from domain D_i
- goal test is a set of constraints specifying allowable combinations of values for subsets of variables

- Examples:
  - Sudoku



5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

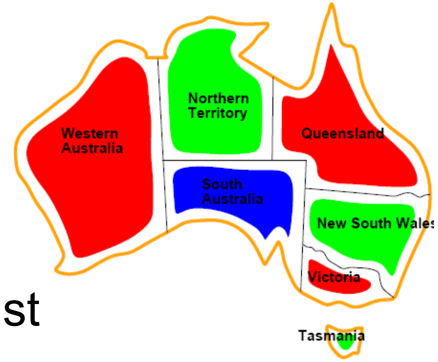
5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

  - cryptarithmetic puzzle

```
   SEND
 + MORE
--------
  MONEY
```

  - n-queens
  - Graph/Map-Coloring

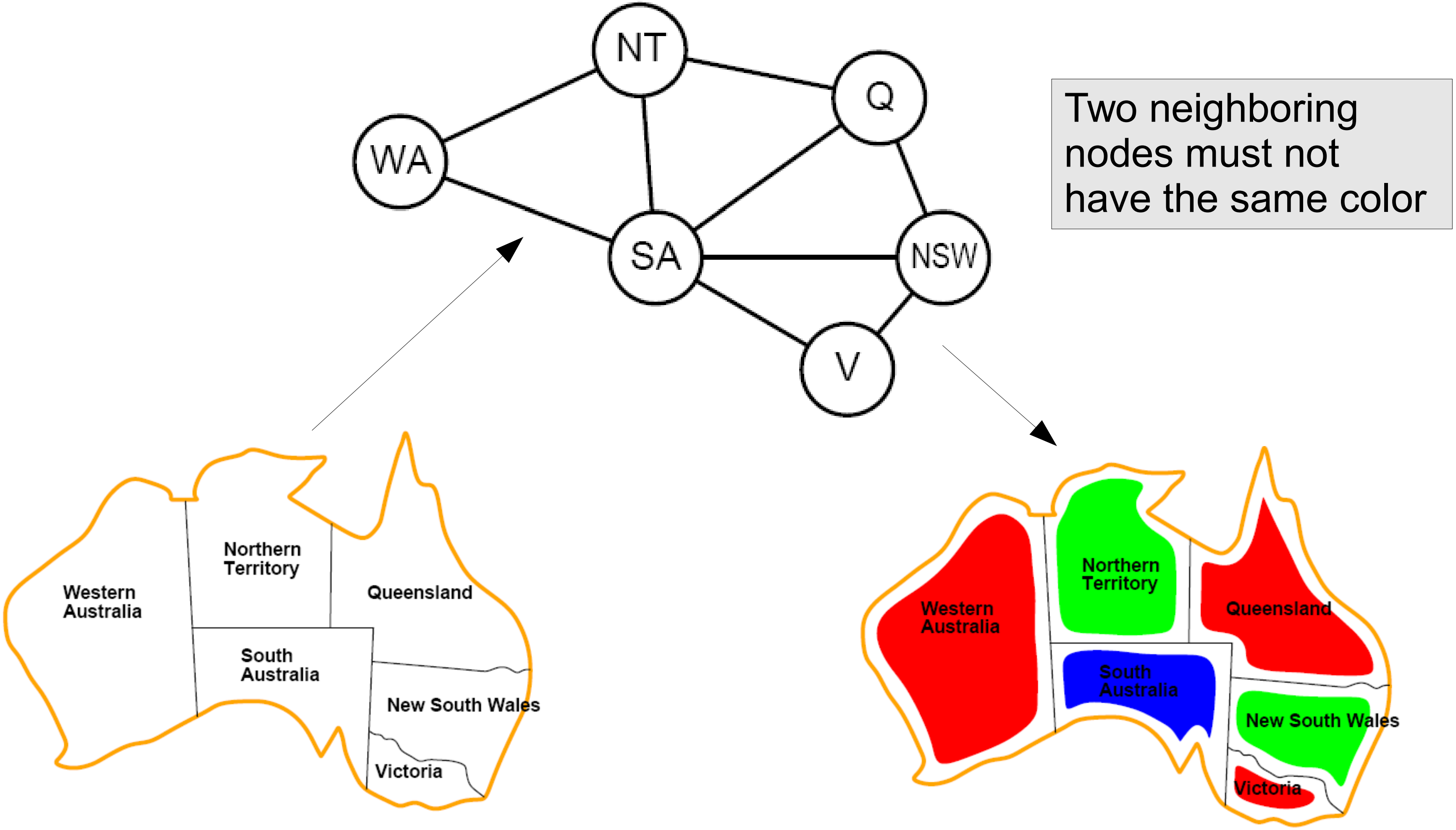
only n colors, neighbors must be different



- Real-world:
  - assignment problems
  - timetables
    - classes, lecturers rooms, studies
  - ...

Constraint Graph

- nodes are variables
- edges indicate constraints between them

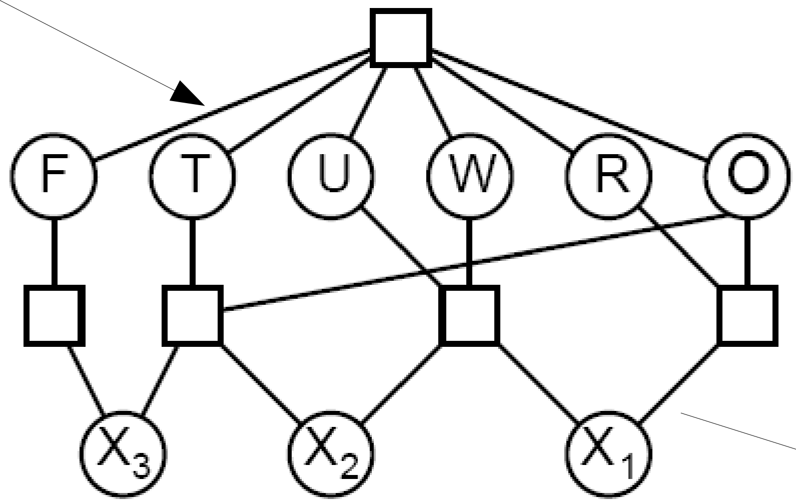


Two neighboring nodes must not have the same color

Constraint Graph

- nodes are variables
- edges indicate constraints between them

$$\begin{array}{r} \mathrm{T\ W\ O} \\ +\ \mathrm{T\ W\ O} \\ \hline \mathrm{F\ O\ U\ R} \end{array}$$



Connected nodes are involved in (in-)equations:

$$2 \cdot O = 10 \cdot X_1 + R$$
$$2 \cdot W + X_1 = 10 \cdot X_2 + U$$
$$2 \cdot T + X_2 = 10 \cdot X_3 + O$$
$$F = X_3$$
$$F \neq T \neq U \neq W \neq R \neq O$$

$$\begin{array}{r} 7\ 3\ 4 \\ +\ 7\ 3\ 4 \\ \hline 1\ 4\ 6\ 8 \end{array}$$

Types of Constraints

- Unary constraints involve a single variable,
  - e.g., *South Australia* $\neq$ *green*
- Binary constraints involve pairs of variables,
  - e.g., *South Australia* $\neq$ *Western Australia*
- Higher-order constraints involve 3 or more variables
  - e.g., $2 \cdot W + X_1 = 10 \cdot X_2 + U$
- Preferences (soft constraints)
  - e.g., *red is better than green*
  - are not binding, but task is to respect as many as possible

→ constrained optimization problems

Solving CSP Problems

Two principal approaches:

- Constraint Propagation:
  - maintain a set of possible values D_i for each variable X_i
  - try to reduce the size of D_i by identifying values that violate some constraints
- Search:
  - successively assign values to variable
  - check all constraints
  - if a constraint is violated → backtrack
  - until all variables have assigned values

Constraint Propagation - Sudoku

- Problem
  - CSP with 81 variables
- Constraints
  - some values are assigned in the start (unary constraints)
  - 27 constraints on 9 values that must all be different

    (9 rows, 9 columns, 9 squares)
- Constraint Propagation
  - People often write a list of possible values into empty fields
  - try to successively eliminate values
- Status
  - Automated constraint solvers can solve the hardest puzzles in no time

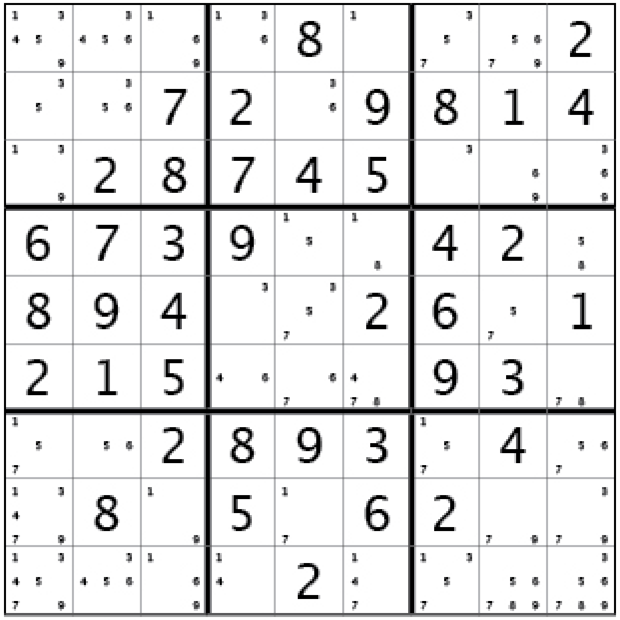


Figure 6

Local Consistency

- make each node in the graph consistent with its neighbors
    - by (iteratively) enforcing the constraints corresponding to the edges

Node Consistency

- the possible values of a variable must conform to all unary constraints
- can be trivially enforced
- Example:
    - Soduko: Some nodes are constrained to a single value

Arc Consistency

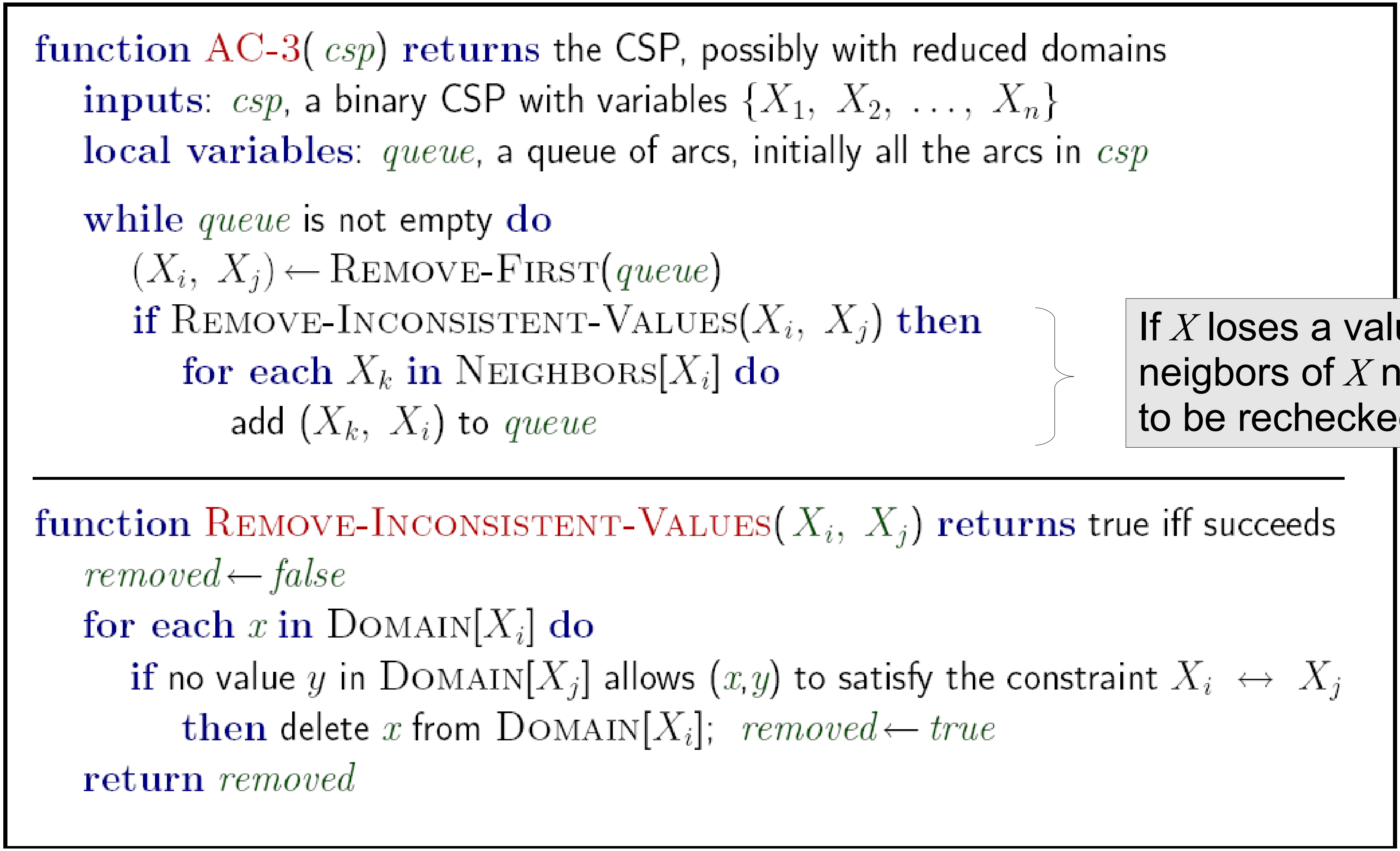
- every domain must be consistent with the neighbors:

A variable X_i is arc-consistent with a variable X_j if

- for every value in its domain D_i
- there is some value in D_j
- that satisfies the constraint on the arc (X_i,X_j)

- can be generalized to n-ary constraints
  - each tuple involving the variable X_i has to be consistent

Arc Consistency Algorithm



```
function AC-3( csp) returns the CSP, possibly with reduced domains
    inputs: csp, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
    local variables: queue, a queue of arcs, initially all the arcs in csp

    while queue is not empty do
        $(X_i, X_j) \leftarrow$ REMOVE-FIRST(queue)
        if REMOVE-INCONSISTENT-VALUES($X_i, X_j$) then
            for each $X_k$ in NEIGHBORS[$X_i$] do
                add $(X_k, X_i)$ to queue
```

```
function REMOVE-INCONSISTENT-VALUES( $X_i, X_j$) returns true iff succeeds
    removed $\leftarrow$ false
    for each $x$ in DOMAIN[$X_i$] do
        if no value $y$ in DOMAIN[$X_j$] allows $(x,y)$ to satisfy the constraint $X_i \leftrightarrow X_j$
            then delete $x$ from DOMAIN[$X_i$];  removed $\leftarrow$ true
    return removed
```

If X loses a value, neigbors of X need to be rechecked.

- Run-time: $O(n^2d^3)$ (can be reduced to $O(n^2d^2)$)
  more efficient than forward checking (see later)

Path Consistency

- Arc Consistency is often sufficient to
  - solve the problem (all domains have size 1)
  - show that the problem cannot be solved (some domains empty)
- but may not be enough
  - there is always a consistent value in the neighboring region

→ Path consistency

  - tightens the binary constraints by considering triples of values

A pair of variables (X_i , X_j) is path-consistent with X_m if

- for every assignment that satisfies the constraint on the arc (X_i,X_j)
- there is an assignment that satisfies the constraints on the arcs (X_i,X_m) and (X_j,X_m)

  - Algorithm AC-3 can be adapted to this case (known as PC-2)

k-Consistency

- The concept can be generalized so that a set of k values need to be consistent

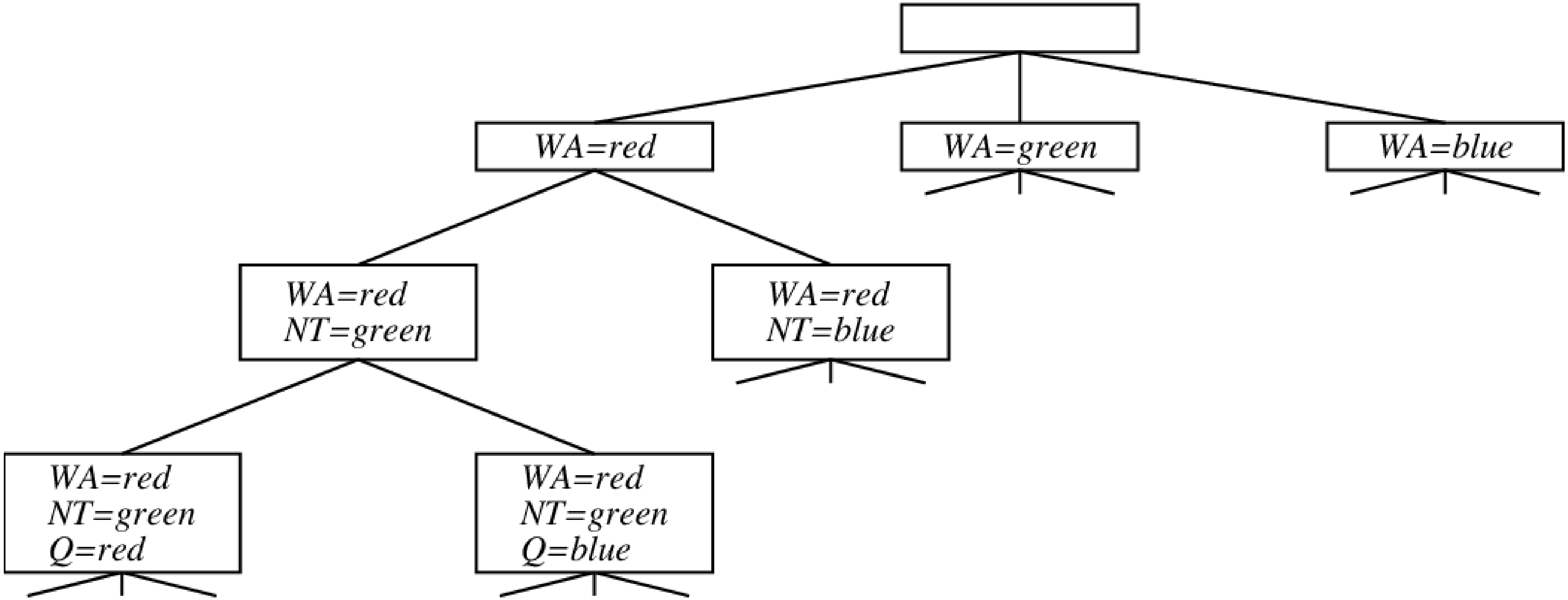
  - 1-consistency = node consistency
  - 2-consistency = arc consistency
  - 3-consistency = path consistency
  - ....

- May lead to faster solution ($O(n^2d)$)
  - but checking for k-Consistency is exponentional in k in the worst case
- therefore arc consistency is most frequently used in practice

Sudoku

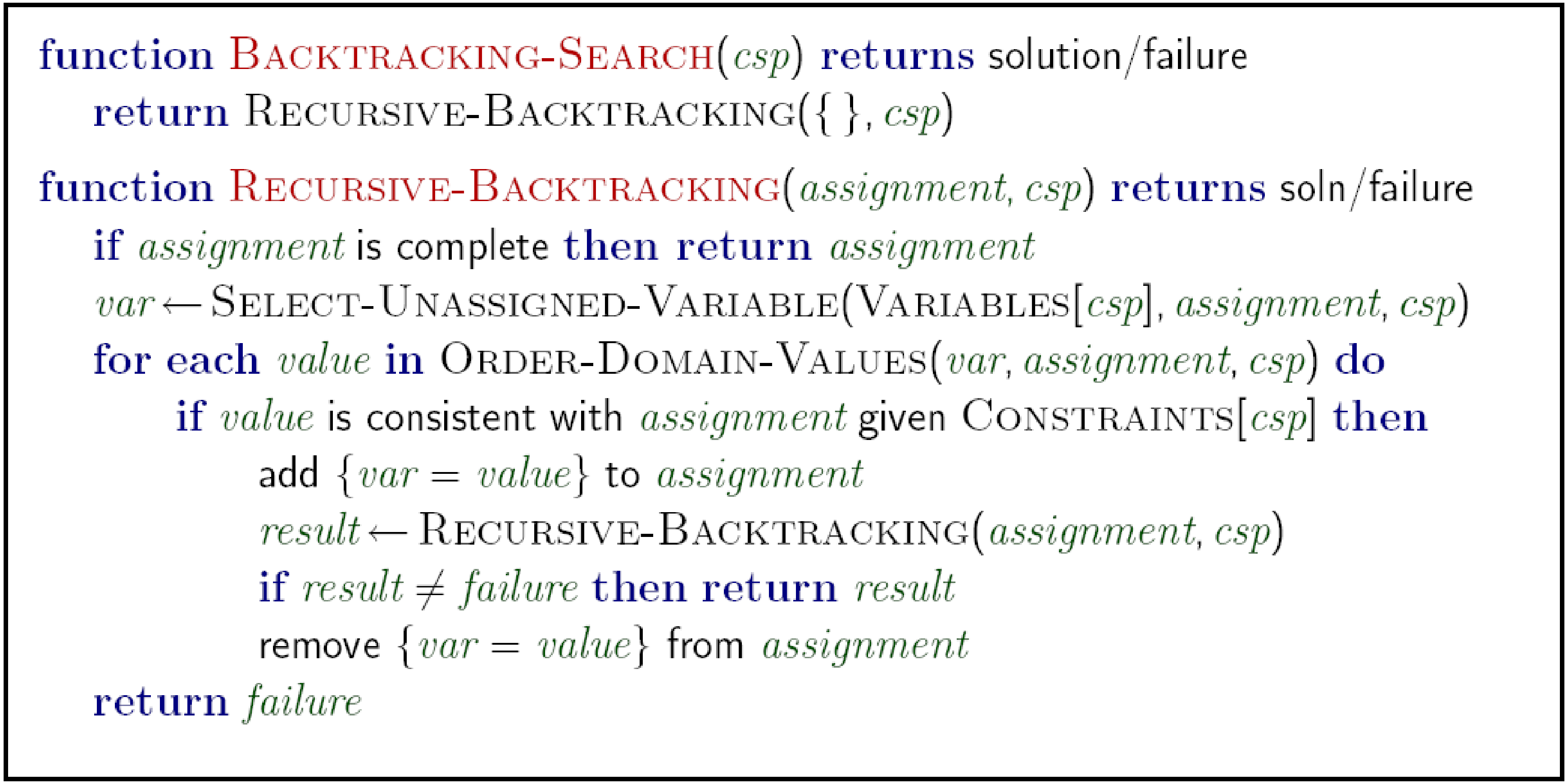
- simple puzzles can be solved with AC-3
  - the 9-valued AllDiff constraints can be converted into pairwise binary constraints (36 each)
  - therefore 27x36 = 972 arc constraints
- somewhat more with PC-2
  - there are 255,960 path constraints
- to solve all puzzles we need a bit of search

Search Tree for CSP



Backtracking Search

- CSP are typically solved with backtracking
  - add one constraint at a time without conflict
  - succeed if a legal assignment is found



```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

Worst-Case Complexity of Backtracking Search

- Assumptions
  - we have n variables

    → all solutions are a depth n in the search tree
  - all variables have v possible values
- Then
  - at level 1 we have $n \cdot v$ possible assignments

    (we can choose one of n variables and one of v values for it)
  - at level 2, we have $(n-1) \cdot v$ possible assignments for each previously assigned variable

    (we can choose one of the remaining $n-1$ variables and one of the v values for it)
  - In general: branching factor at depth l: $(n-l+1) \cdot v$
- Hence
  - The search tree has $n!v^n$ leaves

Note: If the order of variable assigments does not matter the n! may be saved

→ heuristics are needed in SELECT-UNASSIGNED-VARIABLE

General Heuristics for CSP

- Domain-Specific Heuristics
  - Depend on the particular characteristics of the problem
  - Obviously, a heuristic for the 8-puzzle can not be used for the 8-queens problem
- General-purpose heuristics
  - For CSP, good general-purpuse heuristics are known:
  - Mininum Remaining Value Heuristic
    - choose the variable with the fewest consistent values
  - Degree Heuristic
    - choose the variable that imposes the most constraints on the remaining values
  - Least Constraining Value Heuristic
    - Given a variable, choose the value that rules out the fewest values in the remaining variables
  - used in this order, these three can greatly speed up search
    - e.g., n-queens from 25 queens to 1000 queens

SelectUnassignedVariable

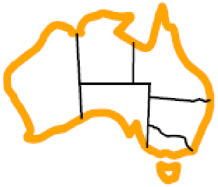
OrderDomainValues

Integrating Constraint Propagation and Backtracking Search

- Performance of Backtracking can be further sped up by integrating constraint propagation into the search
- Key idea:
  - each time a variable is assigned, a constraint propagation algorithm is run in order to reduce the number of choice points in the search
- Possible algorithms
  - Forward Checking
  - AC-3

Forward Checking

- Idea: establish arc consistency for every new variable
  - keep track of remaining legal values for unassigned variables
  - terminate search when any variable has no more legal values



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Forward Checking

- Idea:
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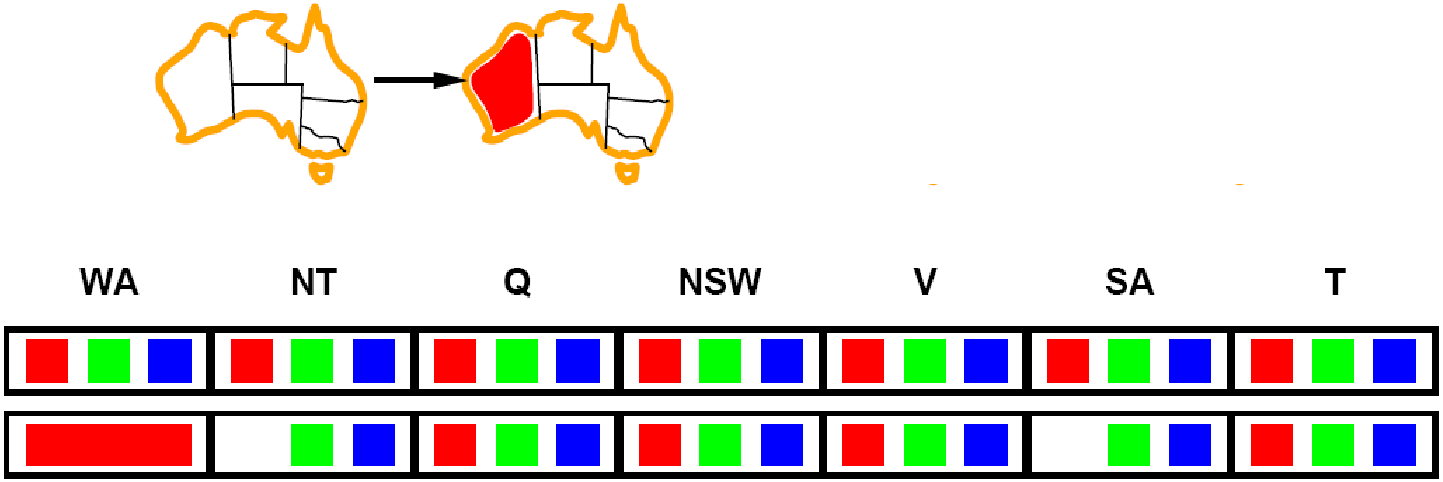
  - terminate search when any variable no legal values



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Forward Checking

- Idea:
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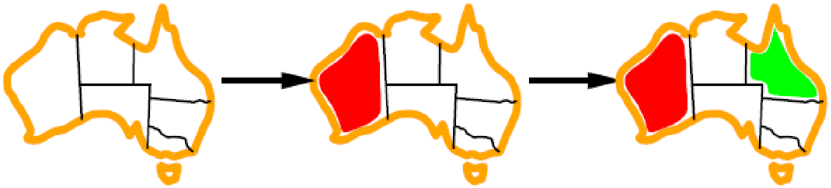

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Forward Checking

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  - keep track of remaining legal values for unassigned variables
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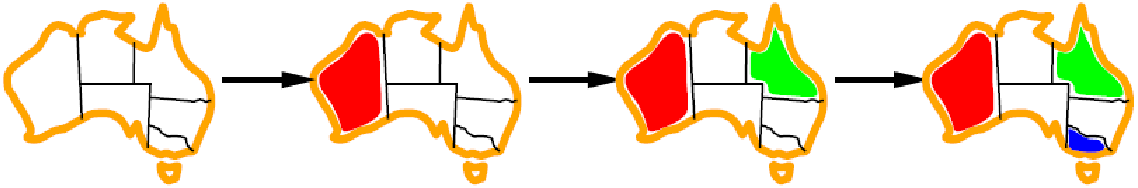

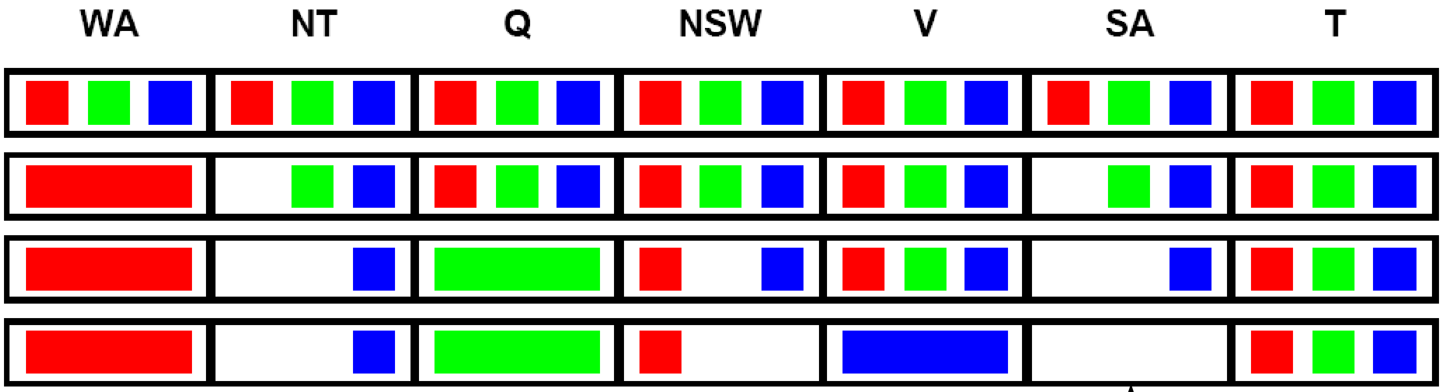

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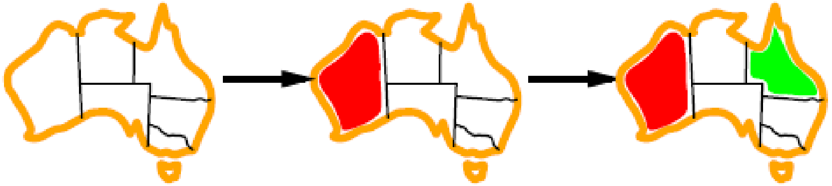
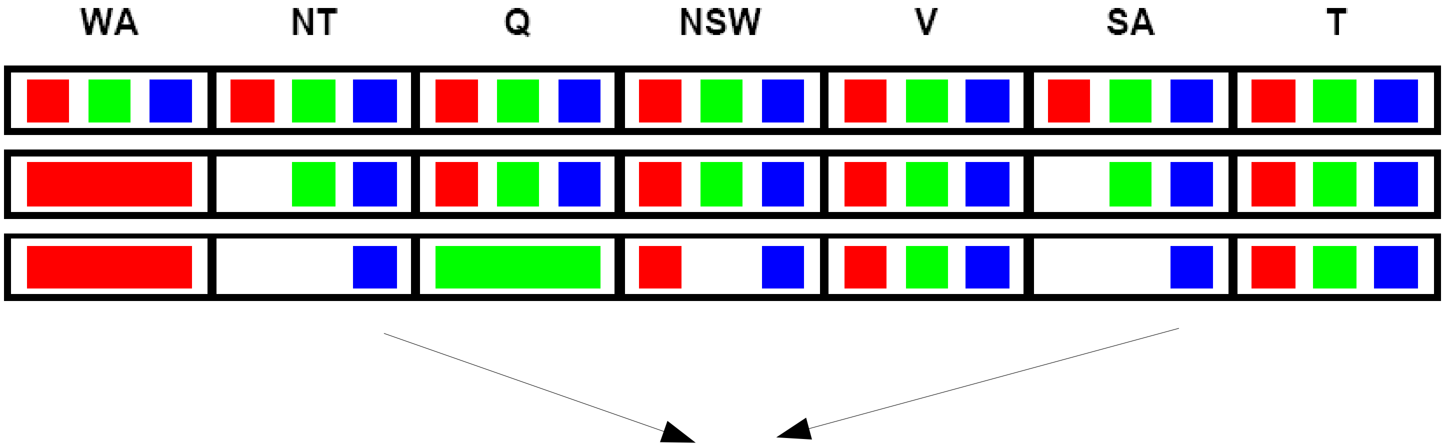
no further assignment possible

Constraint Propagation

- Problem:
  - forward checking propagates information from assigned to unassigned variables
  - but doesn't look ahead to provide early detection for all failures





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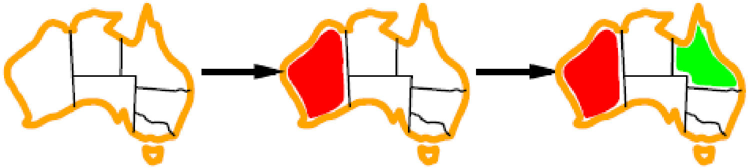
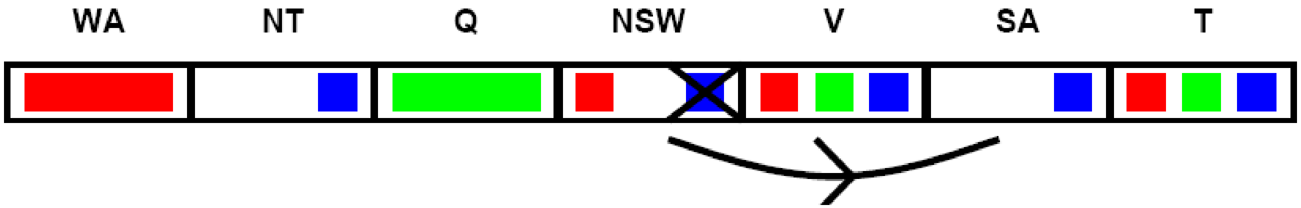
🟥	🟦	🟩	🟥🟦	🟥🟩🟦	🟦	🟥🟩🟦

only one of them can be blue!

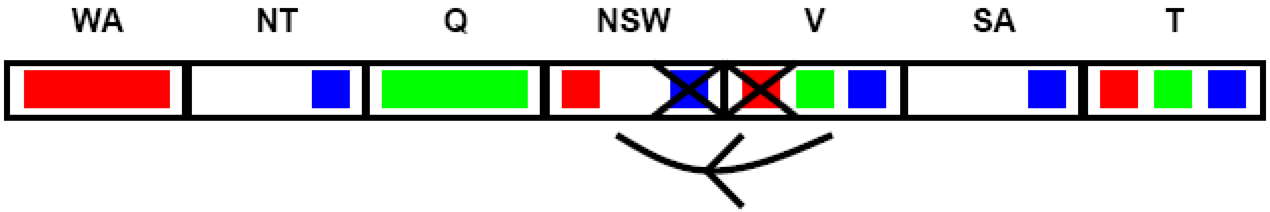
Maintaining Arc Consistency (MAC)

- After each new assignment of a value to a variable, call the AC-3 algorithm but initialize the queue with neighbors



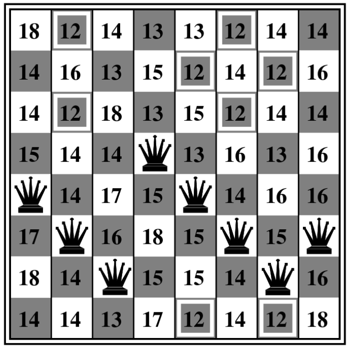


- If one variable (NSW) looses a value (blue), we need to recheck its neighbors as well:



Local Search for CSP

- Modifications for CSPs:
  - work with complete states
  - allow states with unsatisfied constraints
  - operators reassign variable values



18	12	14	13	13	12	14	14
14	16	13	15	12	14	12	16
14	12	18	13	15	12	14	14
15	14	14	♛	13	16	13	16
♛	14	17	15	♛	14	16	16
17	♛	16	18	15	♛	15	♛
18	14	♛	15	15	14	♛	16
14	14	13	17	12	14	12	18

- Min-conflicts Heuristic:
  - randomly select a conflicted variable
  - choose the value that violates the fewest constraints
  - hill-climbing with $h(n) = \#$ of violated constraints

Min-conflicts is the heuristic that we studied for the 8-queens problems.

- Performance:
  - can solve randomly generated CSPs with a high probability
  - except in a narrow range of

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$