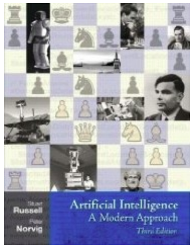
Bayesian Networks

- Syntax
- Semantics
- Parametrized Distributions
- Inference in Bayesian Networks
  - Exact Inference
    - enumeration
    - variable elimination
  - Approximate Inference
    - stochastic simulation
    - Markov Chain Monte Carlo (MCMC)



Many slides based on Russell & Norvig's slides *Artificial Intelligence: A Modern Approach*

Inference Tasks

- Simple queries
  - compute the posterior marginal distribution for a variable
- Conjunctive queries
  - compute the posterior for a conjunction of variables

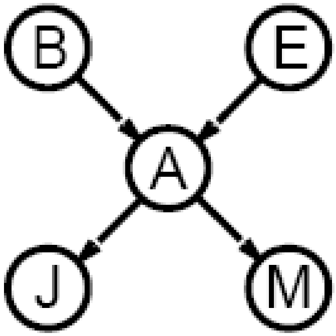
$$\mathbf{P}(X_i, X_j \mid \mathbf{E}=\mathrm{e}) = \mathbf{P}(X_i \mid \mathbf{E}=\mathrm{e}) \cdot \mathbf{P}(X_j \mid X_i, \mathbf{E}=\mathrm{e})$$

- Optimal decisions
  - decision networks include utility information
  - probabilistic inference required for $P(outcome|action,evidence)$
- Value of Information
  - Which evidence to seek next?
- Sensitivity Analysis
  - Which probability values are most critical?
- Explanation
  - Why do I need a new starter motor?

Inference by Enumeration

Slightly intelligent way to sum out variables from the joint without actually constructing its explicit representation

Simple query on the burglary network:



$\mathbf{P}(B|j,m)$
$= \mathbf{P}(B,j,m)/P(j,m)$
$= \alpha\mathbf{P}(B,j,m)$
$= \alpha\ \Sigma_e\ \Sigma_a\ \mathbf{P}(B,e,a,j,m)$

Worst case: $O(n\ d^{\ n})$ time
$O(d^{\ n})$ terms, each consisting of a product of $O(n)$ probabilities

Rewrite full joint entries using product of CPT entries:

$\mathbf{P}(B|j,m)$
$= \alpha\ \Sigma_e\ \Sigma_a\ \mathbf{P}(B)P(e)\mathbf{P}(a|B,e)P(j|a)P(m|a)$
$= \alpha\mathbf{P}(B)\ \Sigma_e\ P(e)\ \Sigma_a\ \mathbf{P}(a|B,e)P(j|a)P(m|a)$

Recursive depth-first enumeration: $O(n)$ space, $O(d^n)$ time

where n is the number of variables and d is the number of values per variable

Enumeration Algorithm

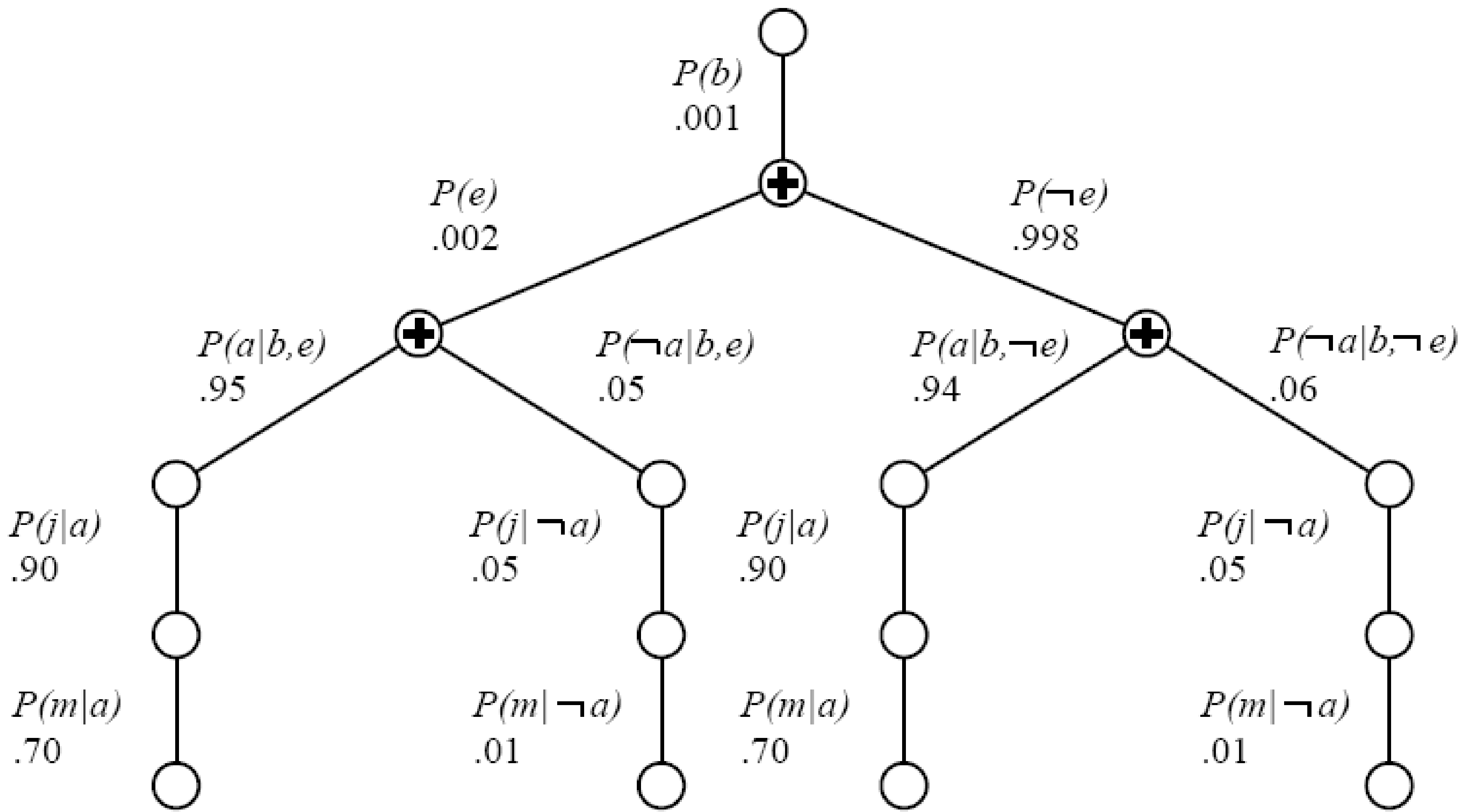
function ENUMERATION-ASK(X, $\mathbf{e}$, bn) **returns** a distribution over X
  inputs: X, the query variable
    $\mathbf{e}$, observed values for variables $\mathbf{E}$
    bn, a Bayesian network with variables $\{X\} \cup \mathbf{E} \cup \mathbf{Y}$

  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    extend $\mathbf{e}$ with value x_i for X
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL(VARS[bn], $\mathbf{e}$)
  return NORMALIZE($\mathbf{Q}(X)$)

---

function ENUMERATE-ALL($vars$, $\mathbf{e}$) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $Y \leftarrow$ FIRST($vars$)
  if Y has value y in $\mathbf{e}$
    then return $P(y \mid Pa(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}$)
    else return $\Sigma_y\ P(y \mid Pa(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}_y$)
      where $\mathbf{e}_y$ is $\mathbf{e}$ extended with $Y = y$

Evaluation Tree



Enumeration is inefficient: repeated computation

e.g., computes $P(j|a)P(m|a)$ for each value of e

Variable Elimination

- Key idea:
  - Do not multiply left-to-right but right-to-left.
  - Thus, terms that appear inside sums are evaluated first
    - intermediate results are stored as so-called factors
    - factors can be re-used several times in the same computation
  - is a form of dynamic programming
- Example: $\mathbf{P}(B|j,m)$

$$= \alpha \underbrace{\mathbf{P}(B)}_{B} \textstyle\sum_e \underbrace{P(e)}_{E} \textstyle\sum_a \underbrace{\mathbf{P}(a|B,e)}_{A} \underbrace{P(j|a)}_{J} \underbrace{P(m|a)}_{M}$$

$$= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) \textstyle\sum_a \mathbf{P}(a|B,e) P(j|a) f_M(a)$$

$$= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) \textstyle\sum_a \mathbf{P}(a|B,e) f_J(a) f_M(a)$$

$$= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) \textstyle\sum_a f_A(a,b,e) f_J(a) f_M(a)$$

$$= \alpha \mathbf{P}(B) \textstyle\sum_e P(e) f_{\bar{A}JM}(b,e) \text{ (sum out } A\text{)}$$

$$= \alpha \mathbf{P}(B) f_{\bar{E}\bar{A}JM}(b) \text{ (sum out } E\text{)}$$

$$= \alpha f_B(b) \times f_{\bar{E}\bar{A}JM}(b)$$

Factors

- A factor is a vector / matrix containing all probabilities for all dependent variables
- Examples:
  - $\mathbf{f}_M(A) = \begin{pmatrix} P(m \mid a) \\ P(m \mid \neg a) \end{pmatrix}$
  - The factor $\mathbf{f}_A(A,B,E)$ is a 2 x 2 x 2 matrix

Basic Operations

- Summing Out a variable from a product of factors
  - move all constant factors outside of the summation
  - add up submatrices in pointwise product of remaining factors

$$\sum_x \mathbf{f}_1 \times \ldots \times \mathbf{f}_k = \mathbf{f}_1 \times \ldots \times \mathbf{f}_i \times \sum_x \mathbf{f}_{i+1} \times \ldots \times \mathbf{f}_k$$
$$= \mathbf{f}_1 \times \ldots \times \mathbf{f}_i \times \mathbf{f}_{\bar{X}}$$

assuming $\mathbf{f}_1$, ..., $\mathbf{f}_i$ do not depend on X

- Pointwise Product of factors $\mathbf{f}_1$ and $\mathbf{f}_2$
  - for example: $\mathbf{f}_1(A,B) \times \mathbf{f}_2(B,C) = \mathbf{f}(A,B,C)$
  - in general: $\mathbf{f}_1(X_1,\ldots,X_j,Y_1,\ldots,Y_k) \times \mathbf{f}_2(Y_1,\ldots,Y_k,Z_1,\ldots,Z_l) =$
    $= \mathbf{f}(X_1,\ldots,X_j,Y_1,\ldots,Y_k,Z_1,\ldots,Z_l)$
    - has 2^{j+k+l} entries (if all variables are binary)

Example: Pointwise Product

p	q	$f_1(p,q)$
T	T	0.1
T	F	0.3
F	T	0.5
F	F	0.7

q	r	$f_2(q,r)$
T	T	0.2
T	F	0.4
F	T	0.6
F	F	0.8

p	q	r	$f_1(p, q)$	$f_2(q, r)$	pointwise product
T	T	T	0.1	0.2	0.1 x 0.2
T	T	F	0.1	0.4	0.1 x 0.4
T	F	T	0.3	0.6	0.3 x 0.6
T	F	F	0.3	0.8	0.3 x 0.8
F	T	T	0.5	0.2	0.5 x 0.2
F	T	F	0.5	0.4	0.5 x 0.4
F	F	T	0.7	0.6	0.7 x 0.6
F	F	F	0.7	0.8	0.7 x 0.8

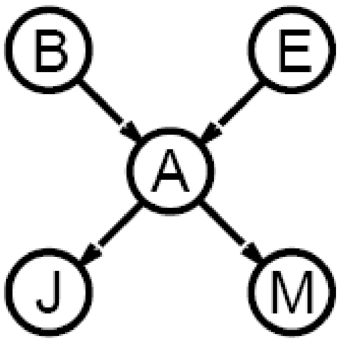
Variable Elimination Algorithm

function ELIMINATION-ASK(X, $\mathbf{e}$, bn) **returns** a distribution over X
  inputs: X, the query variable
    $\mathbf{e}$, evidence specified as an event
    bn, a belief network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

  $factors \leftarrow [\,]$; $vars \leftarrow$ REVERSE(VARS[bn])
  for each var **in** $vars$ **do**
    $factors \leftarrow$ [MAKE-FACTOR(var, $\mathbf{e}$)|$factors$]
    if var is a hidden variable **then** $factors \leftarrow$ SUM-OUT(var, $factors$)
  return NORMALIZE(POINTWISE-PRODUCT($factors$))

Irrelevant Variables

Consider the query $P(JohnCalls|Burglary = true)$



$$P(J|b) = \alpha P(b) \sum_e P(e) \sum_a P(a|b,e) P(J|a) \sum_m P(m|a)$$

Sum over m is identically 1; M is **irrelevant** to the query

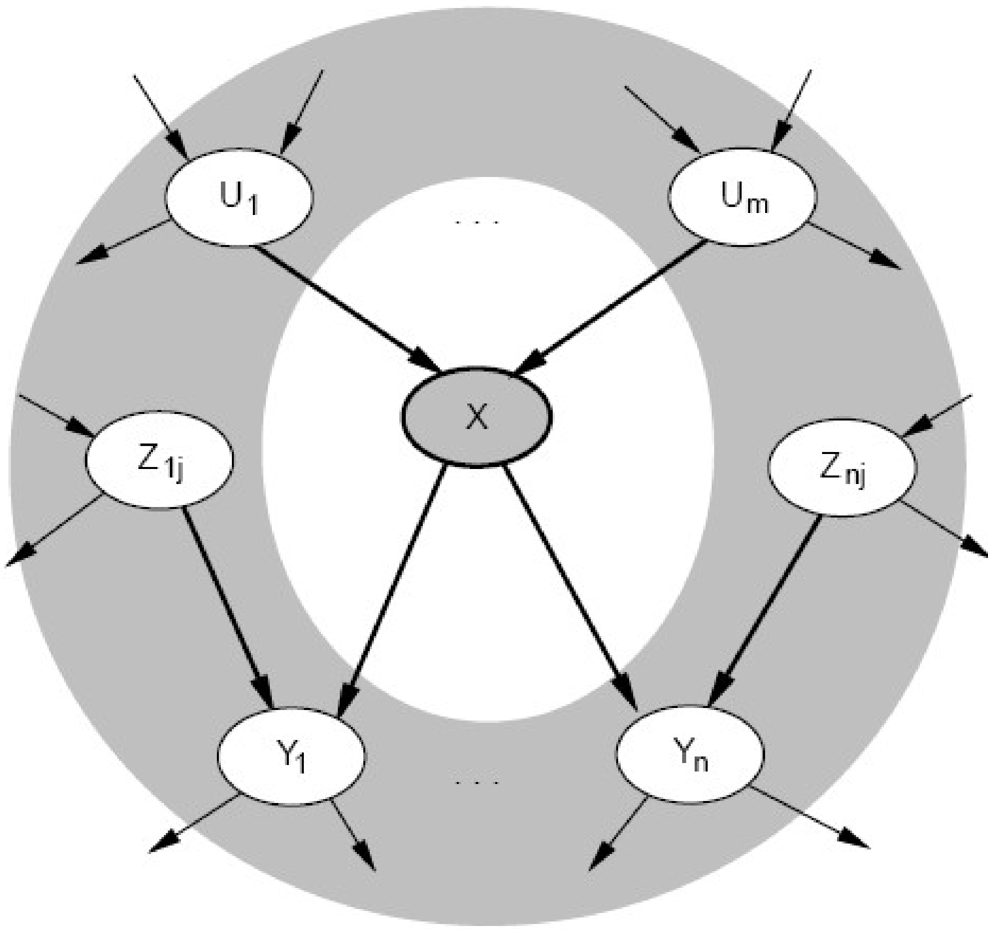
Thm 1: Y is irrelevant unless $Y \in Ancestors(\{X\} \cup \mathbf{E})$

Here, $X = JohnCalls$, $\mathbf{E} = \{Burglary\}$, and $Ancestors(\{X\} \cup \mathbf{E}) = \{Alarm, Earthquake\}$ so $MaryCalls$ is irrelevant

Note: This is similar to backward chaining from a query in Prolog

Markov Blanket

- **Markov Blanket:**
  - parents + children + children's parents

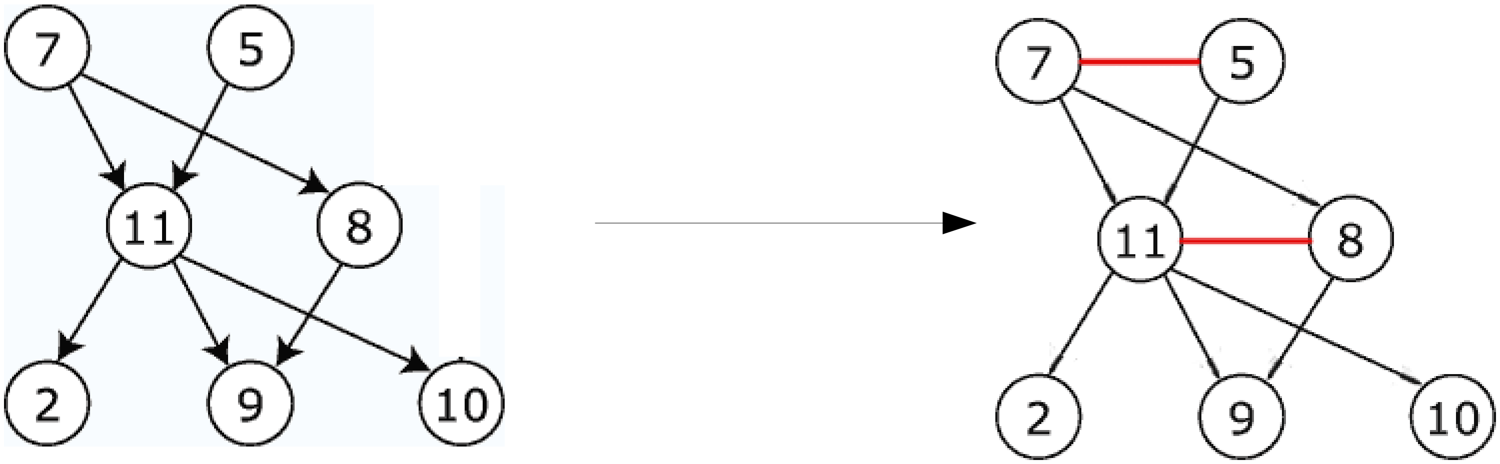


- Each node is conditionally independent of all other nodes given its markov blanket

$$\mathbf{P}(X \mid U_{1,} \ldots, U_m, Y_{1,} \ldots, Y_n, Z_{1j}, \ldots, Z_{nj}) = \\ = \mathbf{P}(X \mid \mathit{all\ variables})$$

Moral Graph

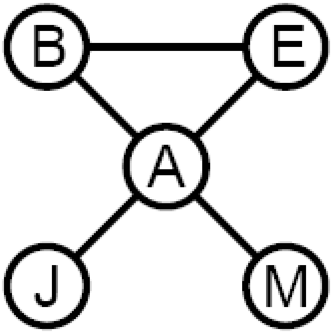
- The moral graph is an undirected graph that is obtained as follows:
  - connect all parents of all nodes
  - make all directed links undirected
- Note:
  - the moral graph connects each node to all nodes of its Markov blanket
    - it is already connected to parents and children
    - now it is also connected to the parents of its children



Graphs taken from Wikipedia

Moral Graph and Irrelevant Variables

- m-separation:
  - A is m-separated from B by C iff it is separated by C in the moral graph
- Example:
  - J is m-separated from E by A



Theorem 2: Y is irrelevant if it is m-separated from X by E

- Example:

For $P(JohnCalls|Alarm = true)$, both $Burglary$ and $Earthquake$ are irrelevant

Complexity of Exact Inference

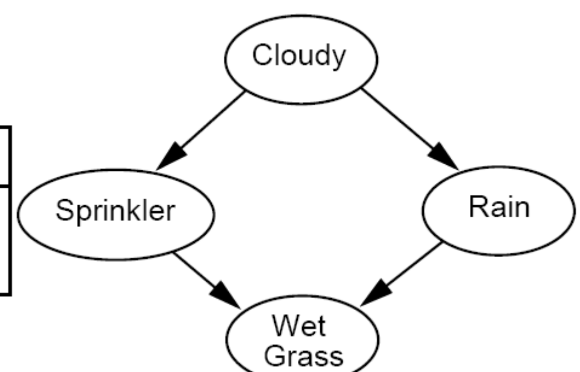
Singly connected networks (or polytrees):

- any two nodes are connected by at most one (undirected) path
- time and space cost of variable elimination are $O(d^k n)$

Multiply connected networks:

- can reduce 3SAT to exact inference $\Rightarrow$ NP-hard

Example:
Two paths from
Cloudy to Wet Grass



P(C)
.50

C	P(S|C)
T	.10
F	.50

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S	R	P(W|S,R)
T	T	.99
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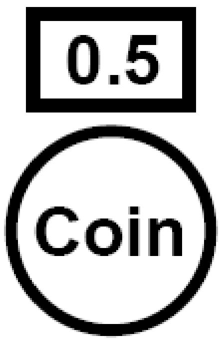
$\rightarrow$ we need approximate inference techniques

Inference by Stochastic Simulation
(Sampling from a Bayesian Network)

Basic idea:

1) Draw N samples from a sampling distribution S
2) Compute an approximate posterior probability $\hat{P}$
3) Show this converges to the true probability P



Outline:

- Sampling from an empty network
- Rejection sampling: reject samples disagreeing with evidence
- Likelihood weighting: use evidence to weight samples
- Markov chain Monte Carlo (MCMC): sample from a stochastic process whose stationary distribution is the true posterior

Sampling from an Empty Network

- Generating samples from a network that has no evidence associated with it (*empty* network)
- Basic idea
  - sample a value for each variable in topological order
  - using the specified conditional probabilities

```
function PRIOR-SAMPLE(bn) returns an event sampled from bn
   inputs: bn, a belief network specifying joint distribution P(X_1, ..., X_n)

   x ← an event with n elements
   for i = 1 to n do
        x_i ← a random sample from P(X_i | parents(X_i))
              given the values of Parents(X_i) in x
   return x
```

Example

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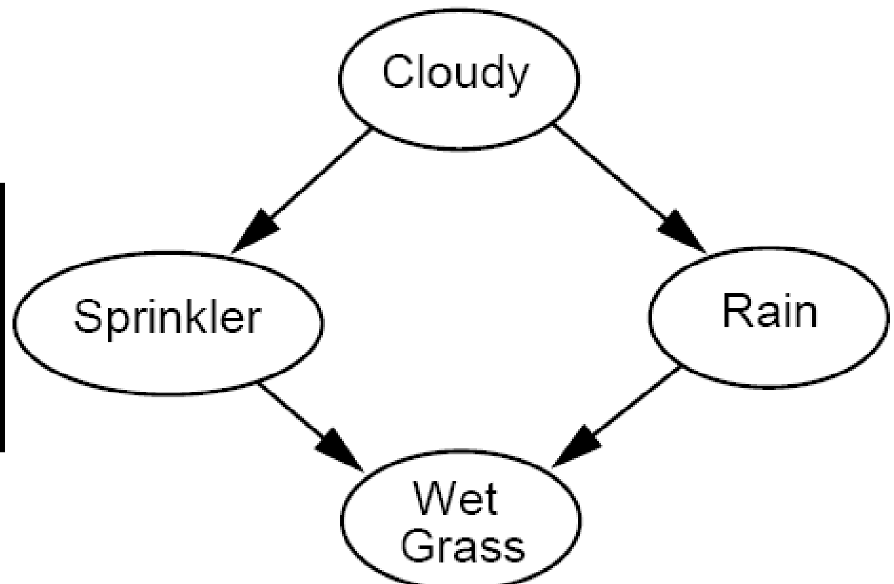
.50



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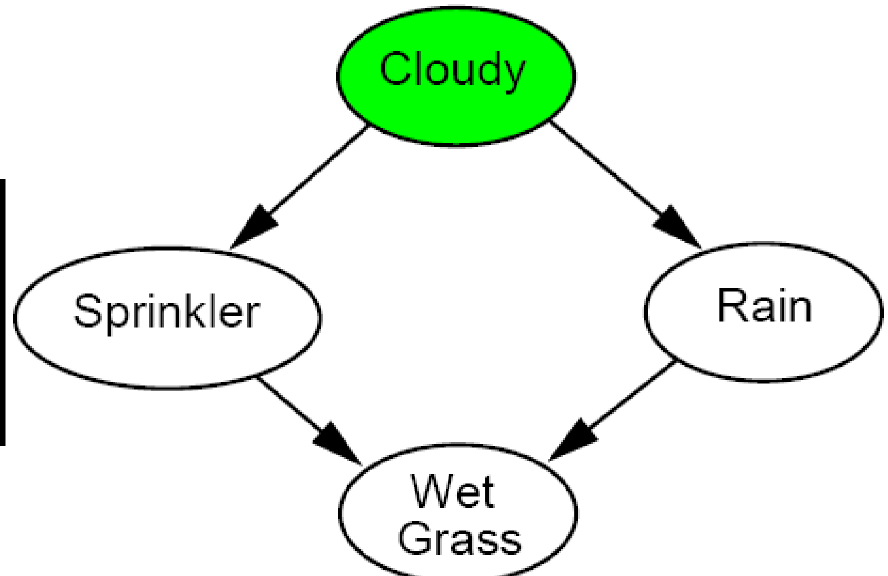

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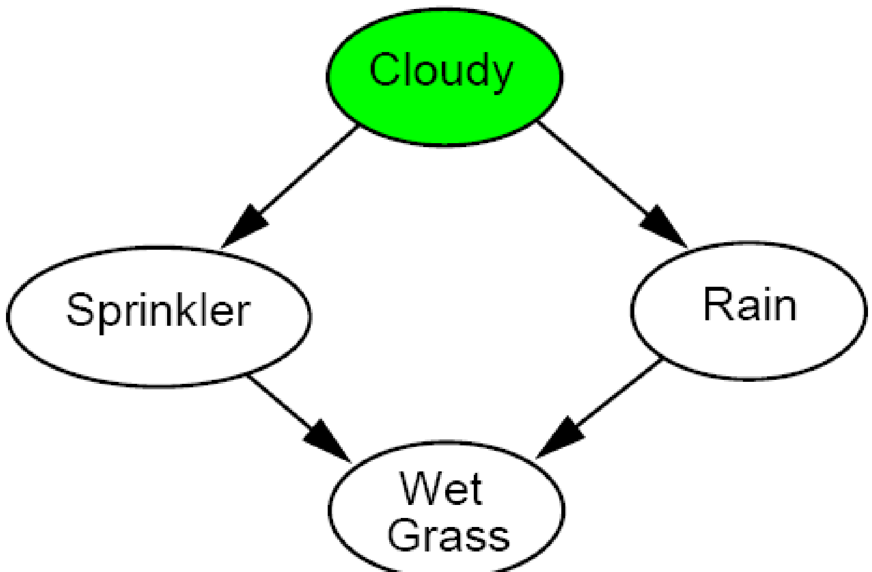

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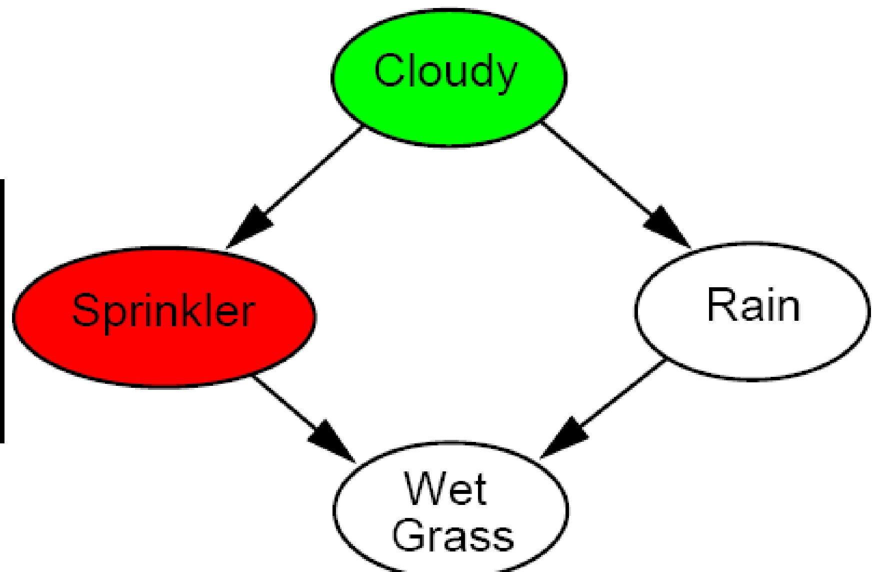

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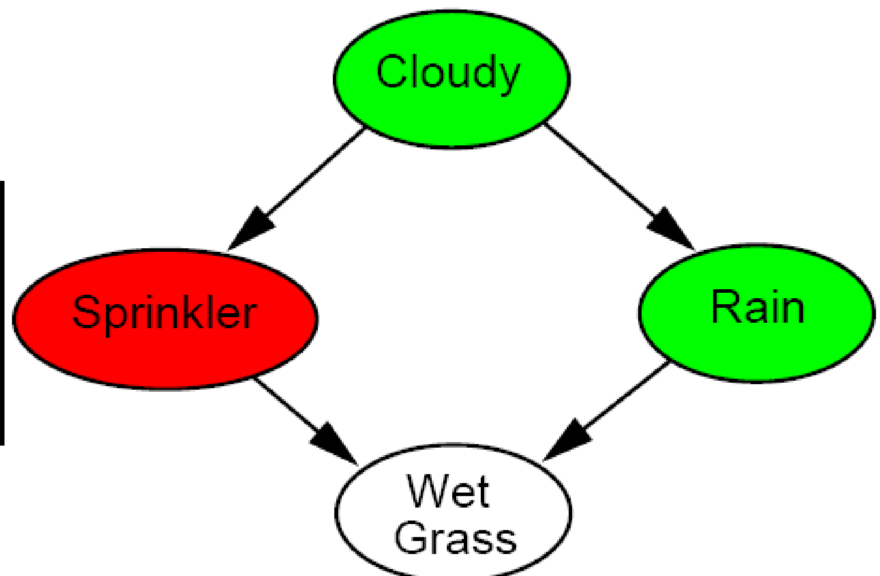

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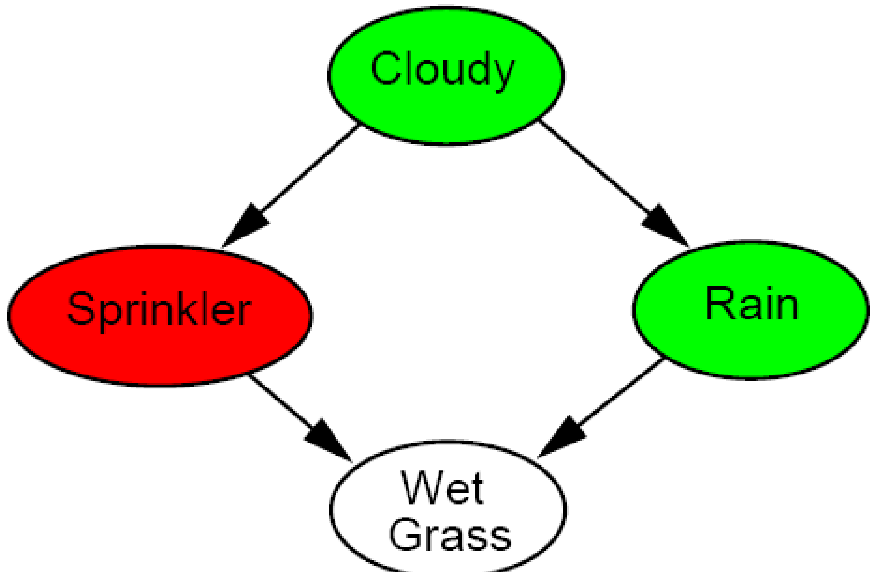

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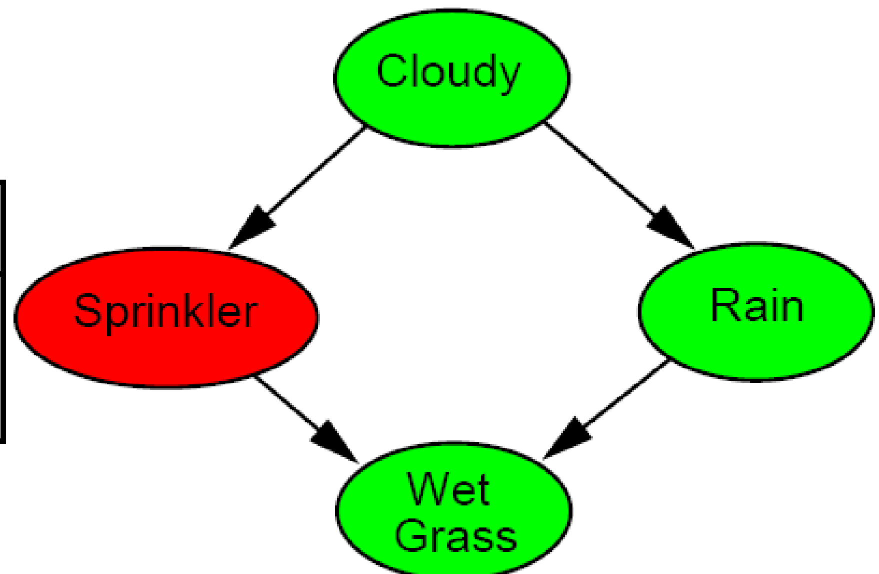

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Probability Estimation using Sampling

- sample many points using the above algorithm
- count how often each possible combination $x_1, x_2, \ldots, x_n$ appears
  - increment counters $N_{PS}(x_1 \ldots x_n)$
- estimate the probability by the observed percentages
  - $\hat{P}_{PS}(x_1 \ldots x_n) = N_{PS}(x_1 \ldots x_n) / N$
- does this converge towards the joint probability function?

Convergence of Sampling from an Empty Network

Probability that PRIORSAMPLE generates a particular event

$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i)) = P(x_1 \ldots x_n)$

i.e., the true prior probability

E.g., $S_{PS}(t, f, t, t) = 0.5 \times 0.9 \times 0.8 \times 0.9 = 0.324 = P(t, f, t, t)$

Let $N_{PS}(x_1 \ldots x_n)$ be the number of samples generated for event $x_1, \ldots, x_n$

Then we have

$$\begin{aligned}
\lim_{N\to\infty} \hat{P}(x_1, \ldots, x_n) &= \lim_{N\to\infty} N_{PS}(x_1, \ldots, x_n)/N \\
&= S_{PS}(x_1, \ldots, x_n) \\
&= P(x_1 \ldots x_n)
\end{aligned}$$

That is, estimates derived from PRIORSAMPLE are consistent

Shorthand: $\hat{P}(x_1, \ldots, x_n) \approx P(x_1 \ldots x_n)$

Rejection Sampling

$\hat{\mathbf{P}}(X|\mathbf{e})$ estimated from samples agreeing with $\mathbf{e}$

function REJECTION-SAMPLING(X, $\mathbf{e}$, bn, N) **returns** an estimate of $P(X|\mathbf{e})$
  local variables: $\mathbf{N}$, a vector of counts over X, initially zero

  for $j = 1$ to N **do**
    $\mathbf{x} \leftarrow$ PRIOR-SAMPLE(bn)
    if $\mathbf{x}$ is consistent with $\mathbf{e}$ **then**
      $\mathbf{N}[x] \leftarrow \mathbf{N}[x]+1$ where x is the value of X in $\mathbf{x}$
  return NORMALIZE($\mathbf{N}[X]$)

E.g., estimate $\mathbf{P}(Rain|Sprinkler = true)$ using 100 samples
  27 samples have $Sprinkler = true$
    Of these, 8 have $Rain = true$ and 19 have $Rain = false$.

$\hat{\mathbf{P}}(Rain|Sprinkler = true) = \text{NORMALIZE}(\langle 8, 19 \rangle) = \langle 0.296, 0.704 \rangle$

Similar to a basic real-world empirical estimation procedure

Analysis of Rejection Sampling

- Rejection sampling generates random samples from an empty network
  - and discards all samples that are inconsistent with the evidence

$$
\begin{aligned}
\hat{\mathbf{P}}(X|\mathbf{e}) &= \alpha \mathbf{N}_{PS}(X,\mathbf{e}) && \text{(algorithm defn.)} \\
&= \mathbf{N}_{PS}(X,\mathbf{e})/N_{PS}(\mathbf{e}) && \text{(normalized by } N_{PS}(\mathbf{e})\text{)} \\
&\approx \mathbf{P}(X,\mathbf{e})/P(\mathbf{e}) && \text{(property of } \textsc{PriorSample}\text{)} \\
&= \mathbf{P}(X|\mathbf{e}) && \text{(defn. of conditional probability)}
\end{aligned}
$$

Hence rejection sampling returns consistent posterior estimates

- Problem
  - many unnecessary samples will be generated if the probability of observing the evidence e is small
  - $P(\mathbf{e})$ will decrease exponentially with increasing numbers of evidence variables!

Likelihood Weighting

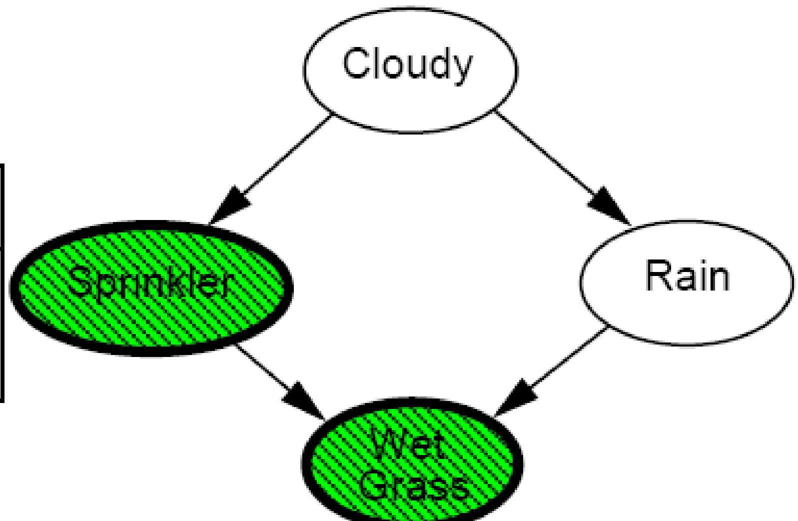
Idea: fix evidence variables, sample only nonevidence variables, and weight each sample by the likelihood it accords the evidence

```
function LIKELIHOOD-WEIGHTING(X, e, bn, N) returns an estimate of P(X|e)
   local variables: W, a vector of weighted counts over X, initially zero

   for j = 1 to N do
       x, w ← WEIGHTED-SAMPLE(bn)
       W[x] ← W[x] + w where x is the value of X in x
   return NORMALIZE(W[X])
```

```
function WEIGHTED-SAMPLE(bn, e) returns an event and a weight

   x ← an event with n elements; w ← 1
   for i = 1 to n do
       if X_i has a value x_i in e
           then w ← w × P(X_i = x_i | parents(X_i))
           else x_i ← a random sample from P(X_i | parents(X_i))
   return x, w
```

Example

Evidence:
sprinkler is on and
grass is wet



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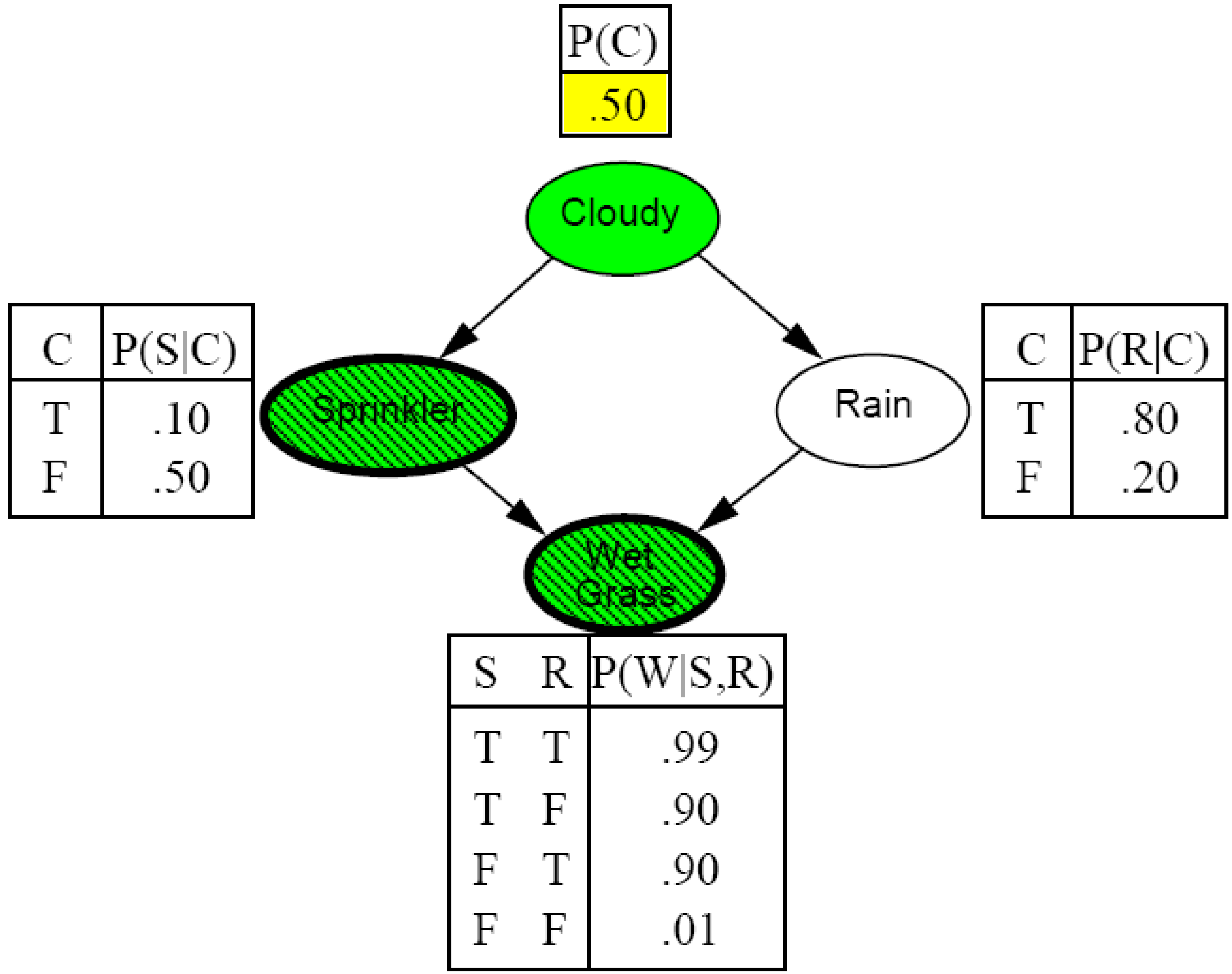
$w = 1.0$

Example



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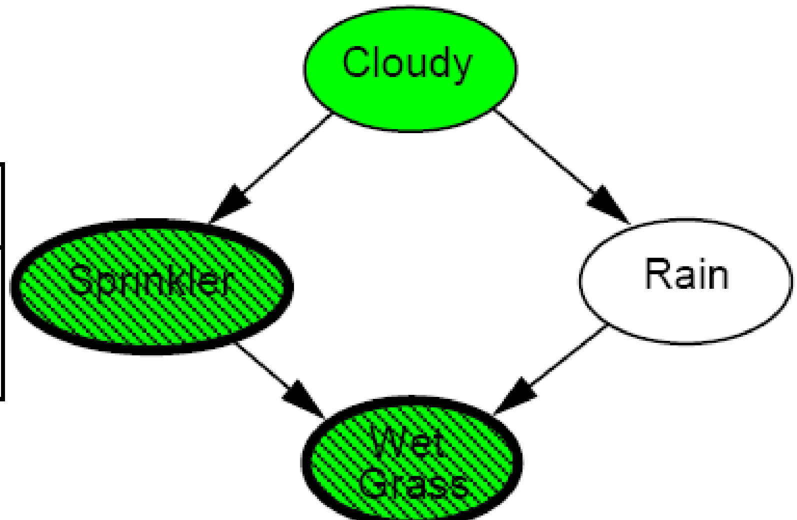

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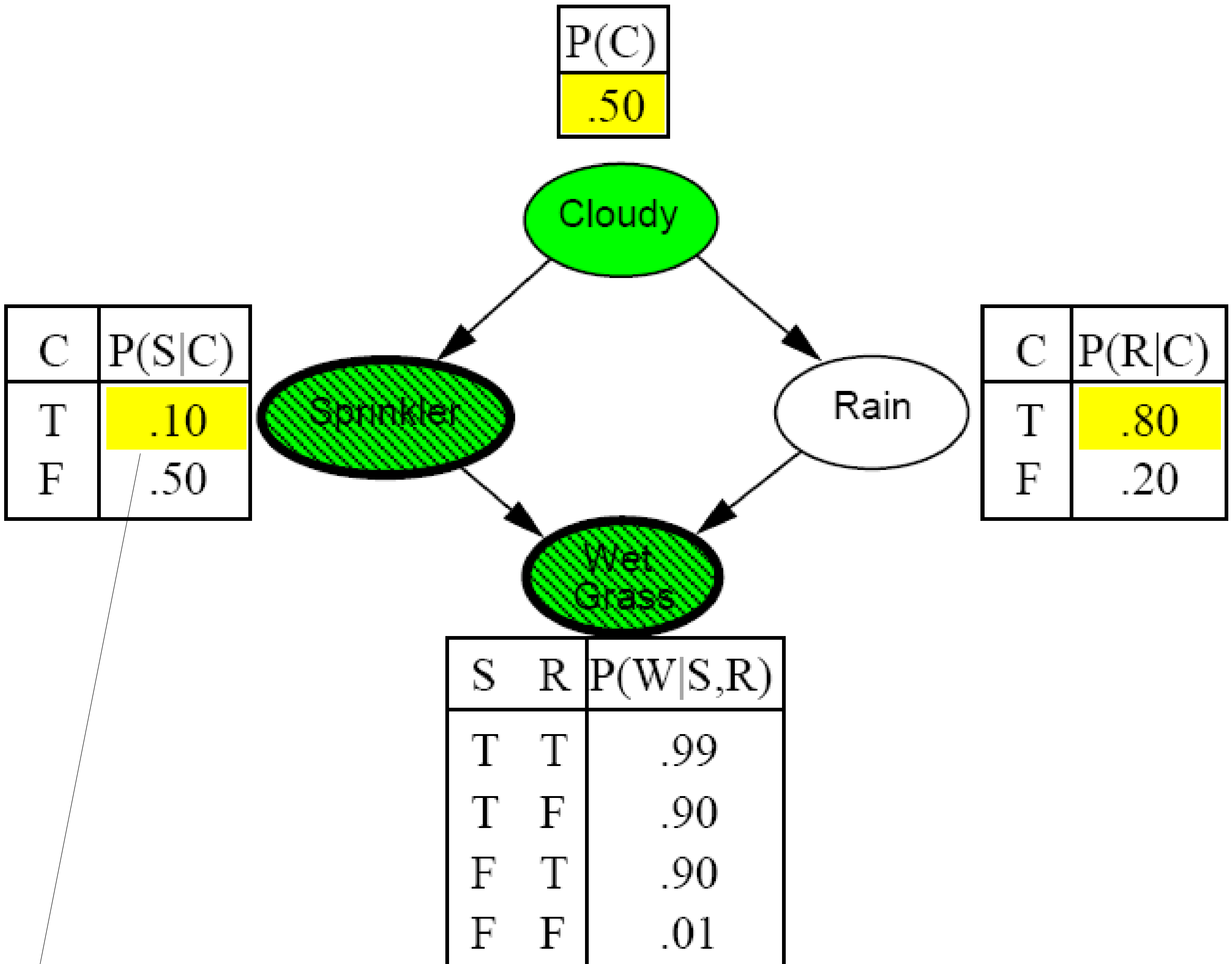
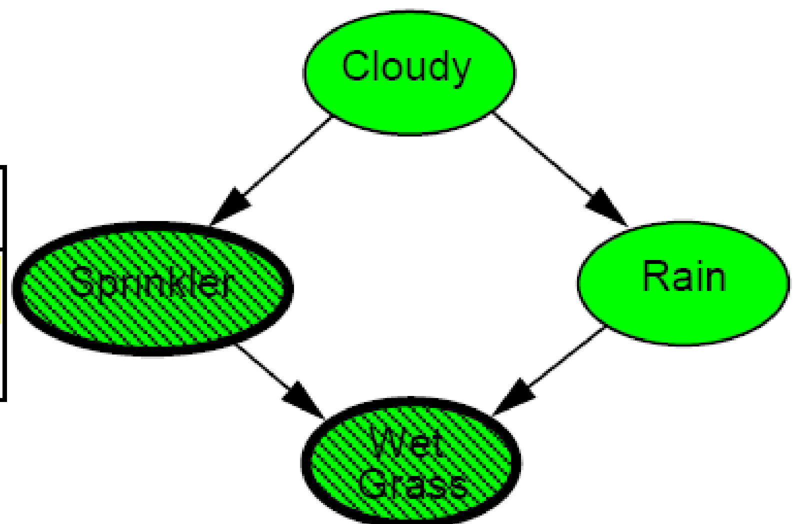


$$w = 1.0 \times 0.1$$

Example



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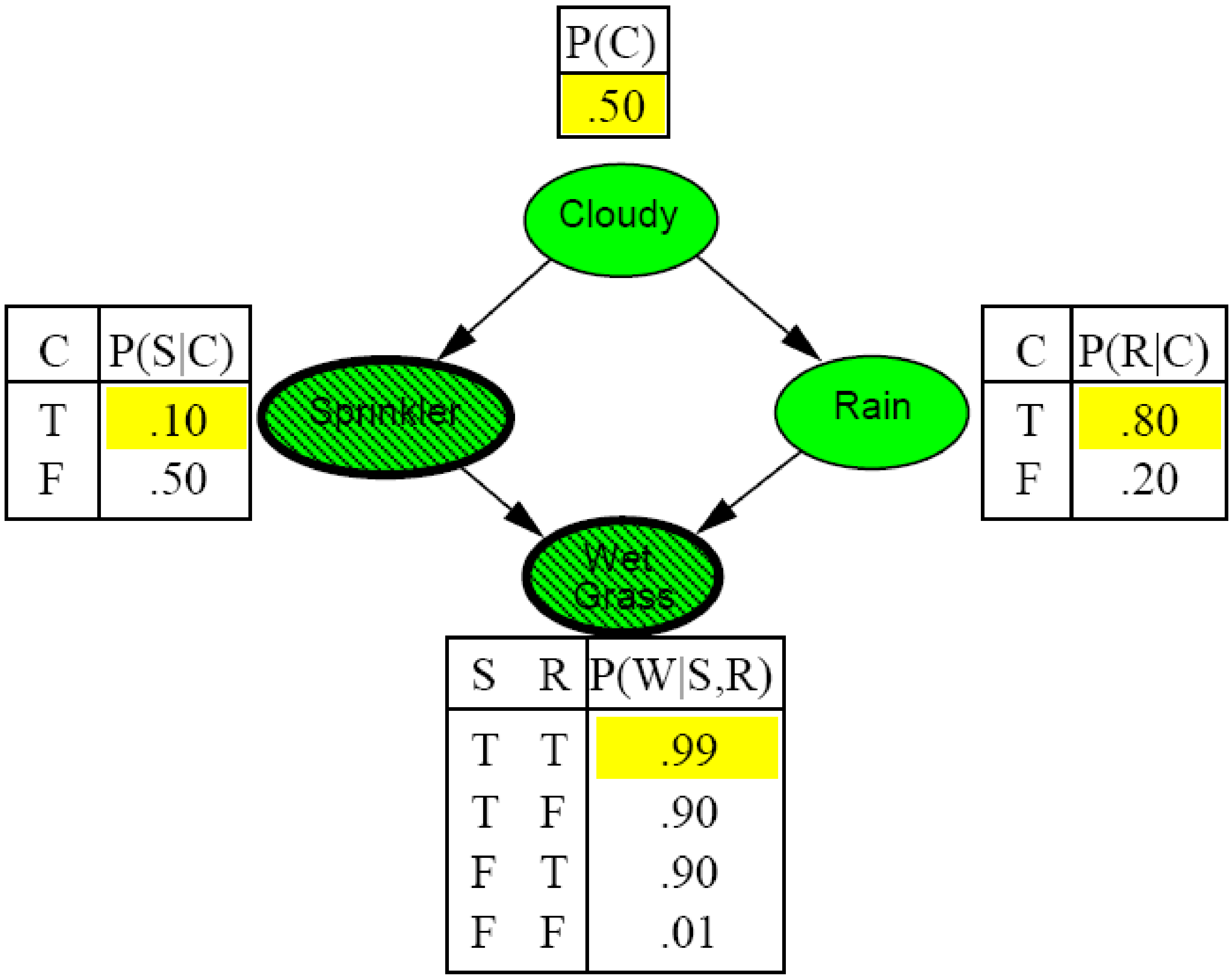
$$w = 1.0 \times 0.1$$

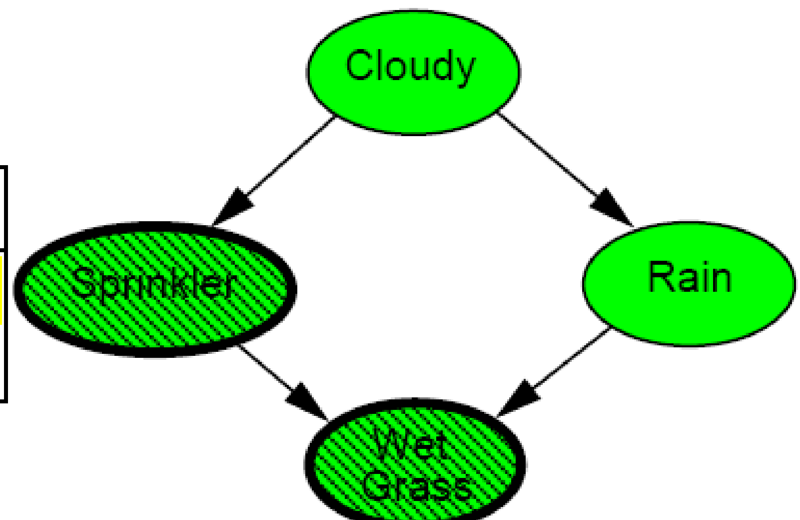
Example



$w = 1.0 \times 0.1$

Example

P(C)
.50



C	P(S|C)
T	.10
F	.50

C	P(R|C)
T	.80
F	.20

S	R	P(W|S,R)
T	T	.99
T	F	.90
F	T	.90
F	F	.01

$$w = 1.0 \times 0.1 \times 0.99 = 0.099$$

The probability (weight) of the event that it is cloudy, the sprinkler is on, it is raining, and the grass is wet is 0.099

Analysis

Sampling probability for WEIGHTEDSAMPLE is

$S_{WS}(\mathbf{z},\mathbf{e}) = \prod_{i=1}^{l} P(z_i|parents(Z_i))$

Note: pays attention to evidence in **ancestors** only

$\Rightarrow$ somewhere "in between" prior and posterior distribution

Weight for a given sample $\mathbf{z},\mathbf{e}$ is

$w(\mathbf{z},\mathbf{e}) = \prod_{i=1}^{m} P(e_i|parents(E_i))$

Weighted sampling probability is

$$S_{WS}(\mathbf{z},\mathbf{e})w(\mathbf{z},\mathbf{e})$$
$$= \prod_{i=1}^{l} P(z_i|parents(Z_i)) \ \prod_{i=1}^{m} P(e_i|parents(E_i))$$
$$= P(\mathbf{z},\mathbf{e}) \text{ (by standard global semantics of network)}$$

Hence likelihood weighting returns consistent estimates but performance still degrades with many evidence variables because a few samples have nearly all the total weight

Markov Chain Monte Carlo (MCMC) Sampling

"State" of network = current assignment to all variables.

Generate next state by sampling one variable given Markov blanket
Sample each variable in turn, keeping evidence fixed

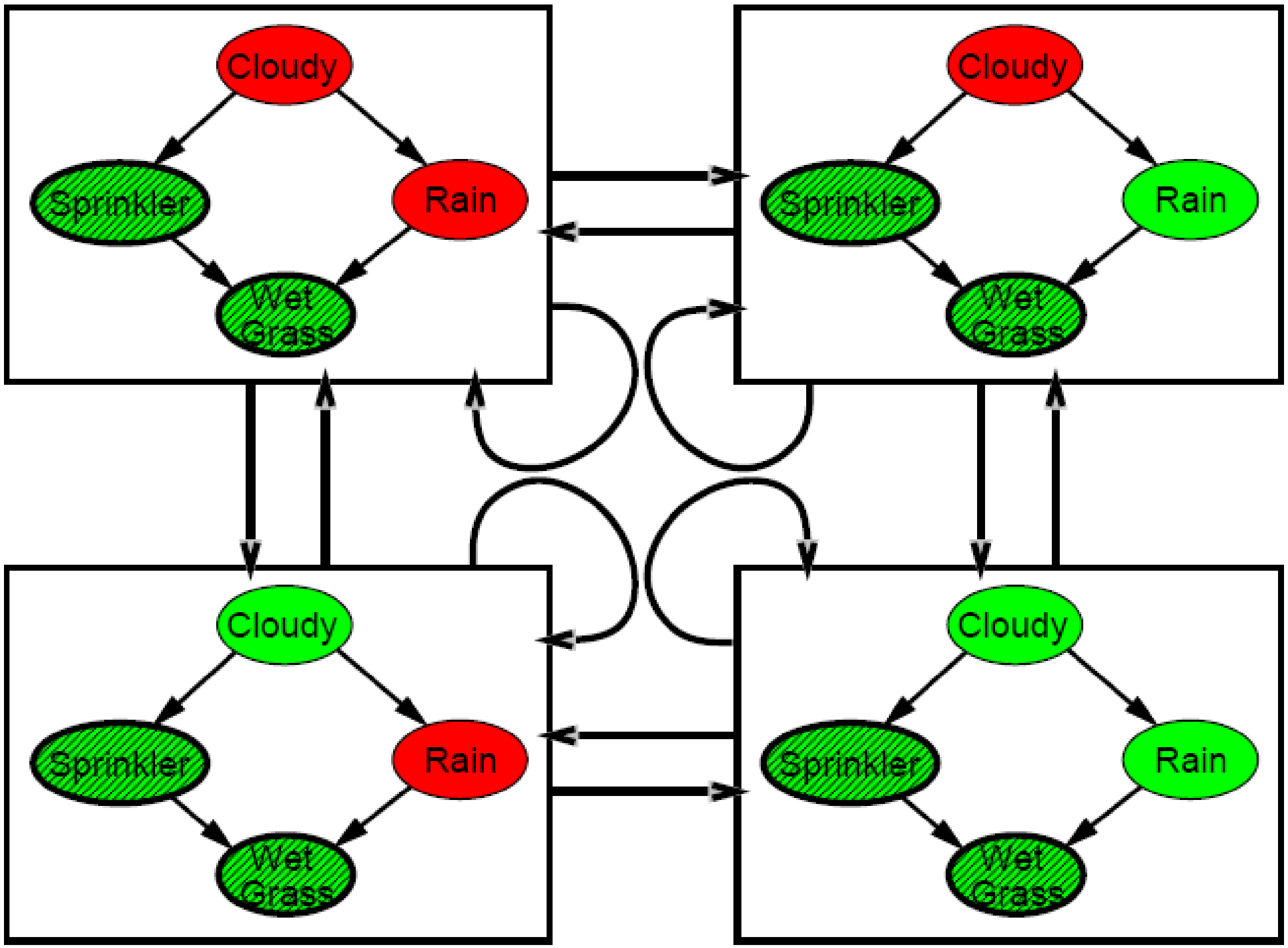
```
function MCMC-Ask(X, e, bn, N) returns an estimate of P(X|e)
   local variables: N[X], a vector of counts over X, initially zero
                    Z, the nonevidence variables in bn
                    x, the current state of the network, initially copied from e

   initialize x with random values for the variables in Y
   for j = 1 to N do
       for each Z_i in Z do
           sample the value of Z_i in x from P(Z_i|mb(Z_i))
               given the values of MB(Z_i) in x
           N[x] ← N[x] + 1 where x is the value of X in x
   return Normalize(N[X])
```

Gibbs Sampling

Can also choose a variable to sample at random each time

The Markov Chain

With $Sprinkler = true, WetGrass = true$, there are four states:



Wander about for a while, average what you see

Example

Estimate $\mathbf{P}(Rain|Sprinkler = true, WetGrass = true)$

Sample $Cloudy$ or $Rain$ given its Markov blanket, repeat.
Count number of times $Rain$ is true and false in the samples.

E.g., visit 100 states
31 have $Rain = true$, 69 have $Rain = false$

$$\hat{\mathbf{P}}(Rain|Sprinkler = true, WetGrass = true)$$
$$= \text{NORMALIZE}(\langle 31, 69 \rangle) = \langle 0.31, 0.69 \rangle$$

Theorem: chain approaches stationary distribution:
long-run fraction of time spent in each state is exactly proportional to its posterior probability