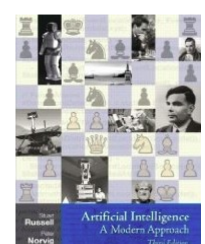
Outline

- Best-first search
  - Greedy best-first search
  - A* search
  - Heuristics
- Local search algorithms
  - Hill-climbing search
  - Beam search
  - Simulated annealing search
  - Genetic algorithms
- Constraint Satisfaction Problems
  - Constraints
  - Constraint Propagation
  - Backtracking Search
  - Local Search



Many slides based on Russell & Norvig's slides *Artificial Intelligence: A Modern Approach*

Constraint Satisfaction Problems

Special Type of search problem:

- state is defined by variables X_i with d values from domain D_i
- goal test is a set of constraints specifying allowable combinations of values for subsets of variables

- Examples:
  - Sudoku

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

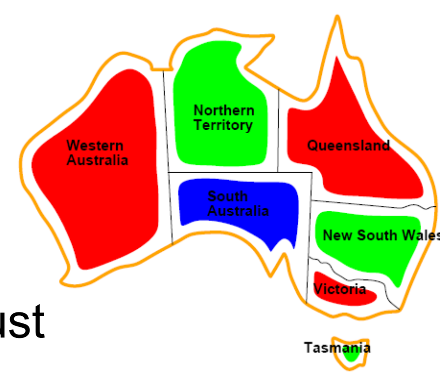
5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

  - cryptarithmetic puzzle

```
   SEND
+  MORE
--------
  MONEY
```

  - n-queens
  - Graph/Map-Coloring

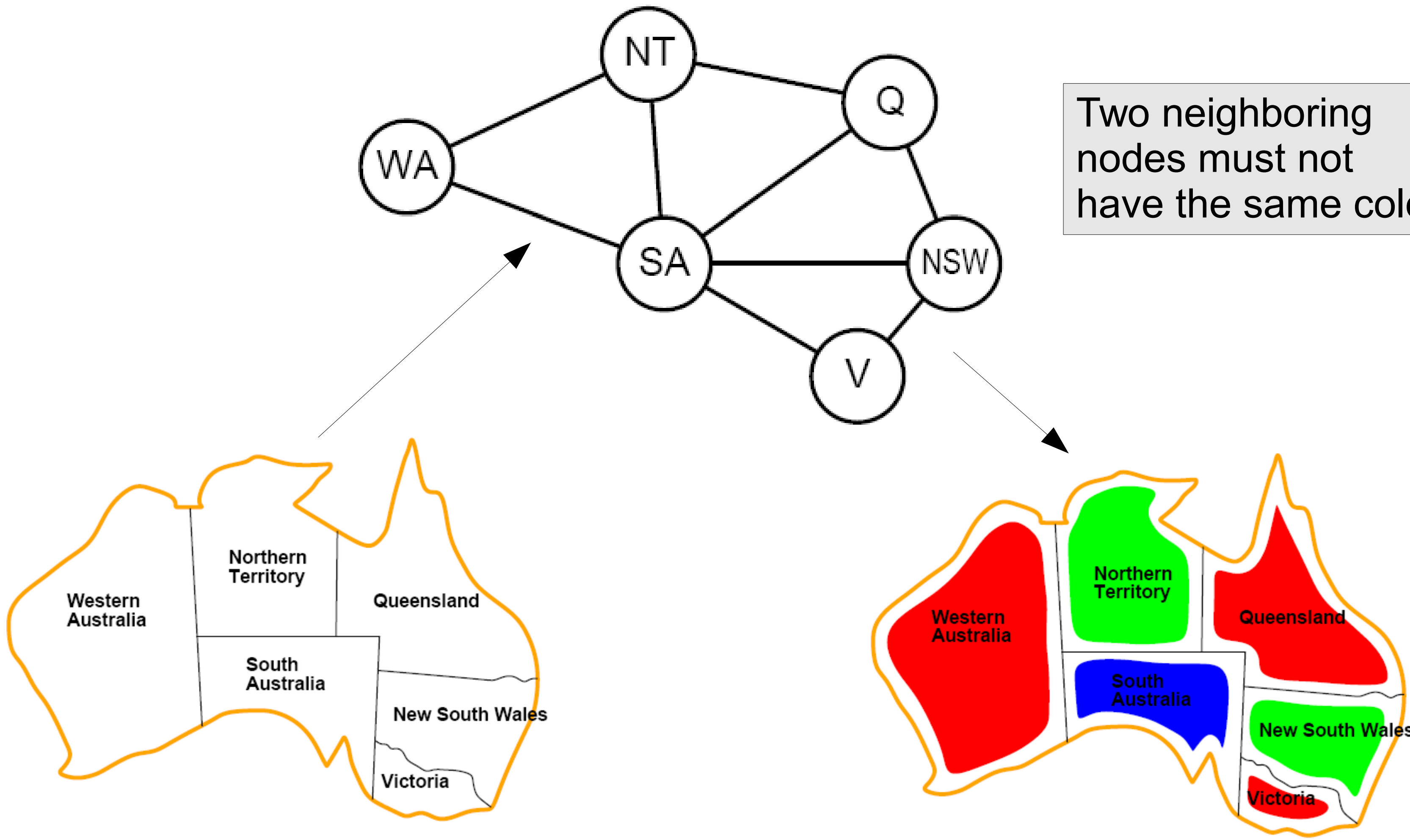
only n colors, neighbors must be different



- Real-world:
  - assignment problems
  - timetables
    - classes, lecturers rooms, studies
  - ...

Constraint Graph

- nodes are variables
- edges indicate constraints between them

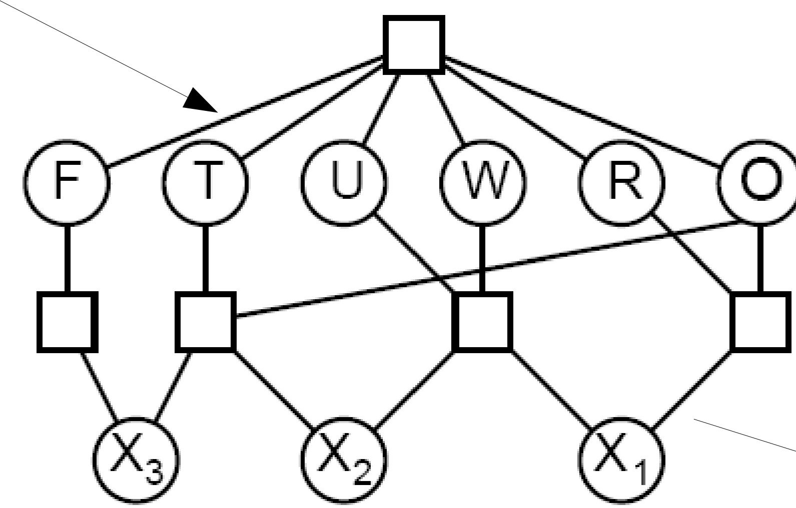


Two neighboring nodes must not have the same color

Constraint Graph

- nodes are variables
- edges indicate constraints between them

$$
\begin{array}{r}
T\;W\;O \\
+\;T\;W\;O \\
\hline
F\;O\;U\;R
\end{array}
$$



Connected nodes are involved in (in-)equations:

$$2 \cdot O = 10 \cdot X_1 + R$$
$$2 \cdot W + X_1 = 10 \cdot X_2 + U$$
$$2 \cdot T + X_2 = 10 \cdot X_3 + O$$
$$F = X_3$$
$$F \neq T \neq U \neq W \neq R \neq O$$

$$
\begin{array}{r}
7\;3\;4 \\
+\;7\;3\;4 \\
\hline
1\;4\;6\;8
\end{array}
$$

Types of Constraints

- Unary constraints involve a single variable,
  - e.g., *South Australia* $\neq$ *green*
- Binary constraints involve pairs of variables,
  - e.g., *South Australia* $\neq$ *Western Australia*
- Higher-order constraints involve 3 or more variables
  - e.g., $2 \cdot W + X_1 = 10 \cdot X_2 + U$
- Preferences (soft constraints)
  - e.g., *red is better than green*
  - are not binding, but task is to respect as many as possible

$\rightarrow$ constrained optimization problems

Solving CSP Problems

Two principal approaches:

- Search:
  - successively assign values to variable
  - check all constraints
  - if a constraint is violated → backtrack
  - until all variables have assigned values
- Constraint Propagation:
  - maintain a set of possible values D_i for each variable X_i
  - try to reduce the size of D_i by identifying values that violate some constraints

Solving Constraint Problems with Search

- Constraint problems define a simple search space:
  - The start node is an empty assignment of values to variables
  - Its successors are all possible ways of assigning one value to a variable (depth 1)
  - Their successors are those with 2 variables assigned (depth 2)
  - ….
  - Until at the end all variables have been assigned a value (depth n)
- Goal test:
  - Does a node at depth n satisfy all constraints?
- Observation:
  - All solution nodes will appear at depth $n \rightarrow$ depth-first search is feasible without losing completeness

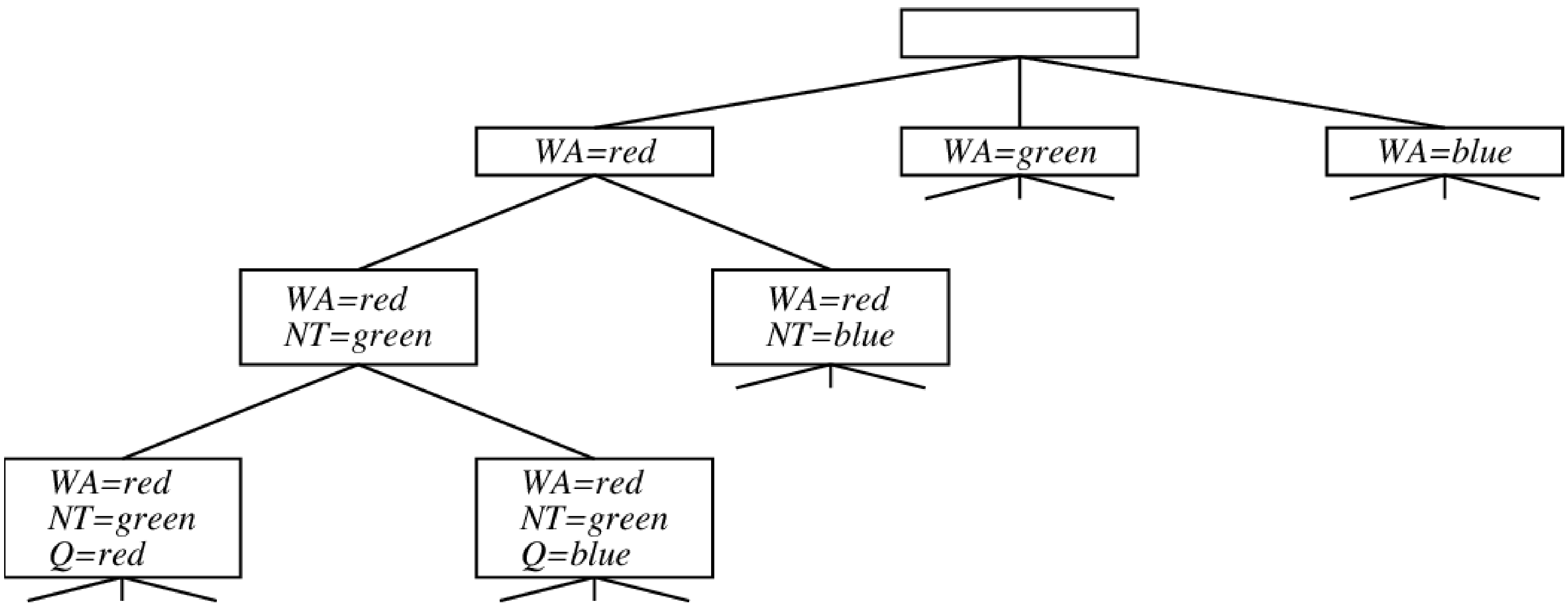
Complexity of Naive Search

- Assumptions
  - we have n variables
    $\rightarrow$ all solutions are a depth n in the search tree
  - all variables have v possible values
- Then
  - at level 1 we have $n \cdot v$ possible assignments
    (we can choose one of n variables and one of v values for it)
  - at level 2, we have $(n-1) \cdot v$ possible assignments for each previously assigned variable
    (we can choose one of the remaining $n-1$ variables and one of the v values for it)
  - In general: branching factor at depth l: $(n-l+1) \cdot v$
- Hence
  - The search tree has $n!v^n$ leaves

Commutative Variable Assignments

- Variable assignments are commutative
  - [$WA = red$ then $NT = green$] is the same as [$NT = green$ then $WA = red$]
- Thus, at each node, we only need to make assignments for one of the variables

  → Total complexity reduces to v^n



Backtracking Search

- Depth-first search with single variable assignments per level is also called backtracking search
- Backtracking is the basic uninformed search algorithm for CSPs
  - add one constraint at a time without conflict
  - succeed if a legal assignment is found
  - Can solve n-queens problems for up to $n \simeq 25$
- Complexity:
  - Worst case is still exponentional
  - heuristics for selecting variables (SELECTUNASSIGNEDVARIABLE) and for ordering values (ORDERDOMAINVALUES) can improve practical performance

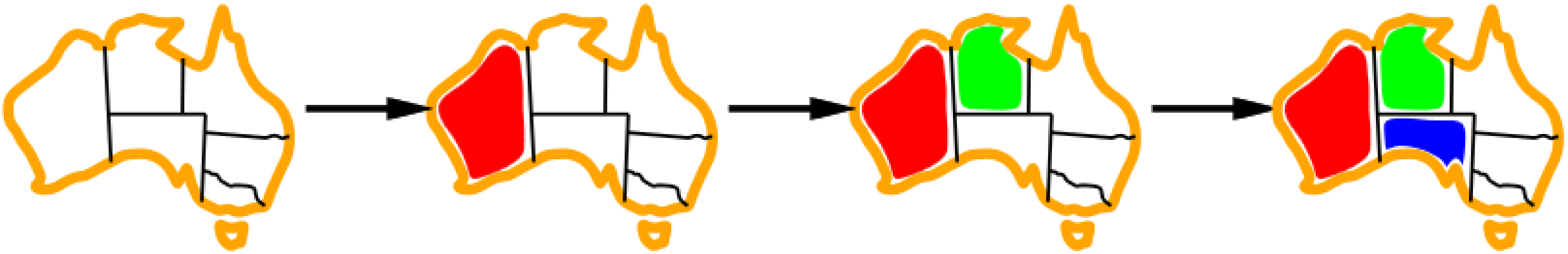
Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

General Heuristics for CSP

- Domain-Specific Heuristics
  - Depend on the particular characteristics of the problem
  - Obviously, a heuristic for the 8-puzzle can not be used for the 8-queens problem
- General-purpose heuristics
  - For CSP, good general-purpuse heuristics are known:
  - Mininum Remaining Values Heuristic
    - choose the variable with the fewest consistent values



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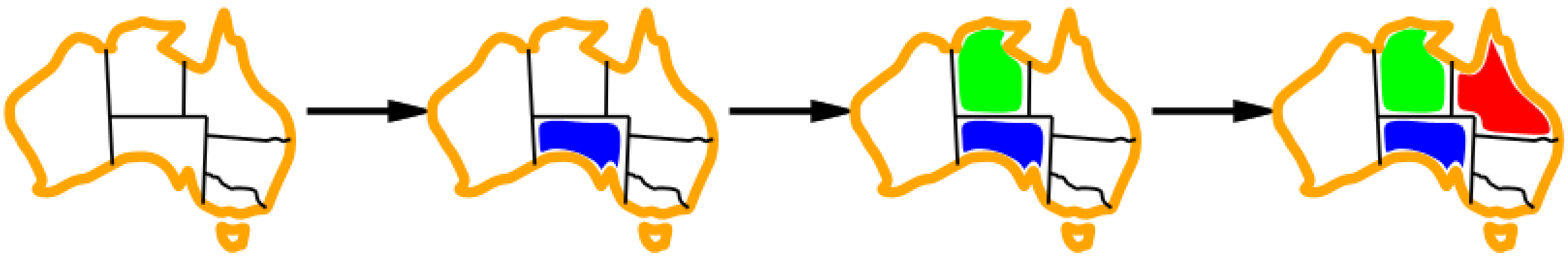
  - Degree Heuristic
    - choose the variable that imposes the most constraints on the remaining values

SelectUnassignedVariable



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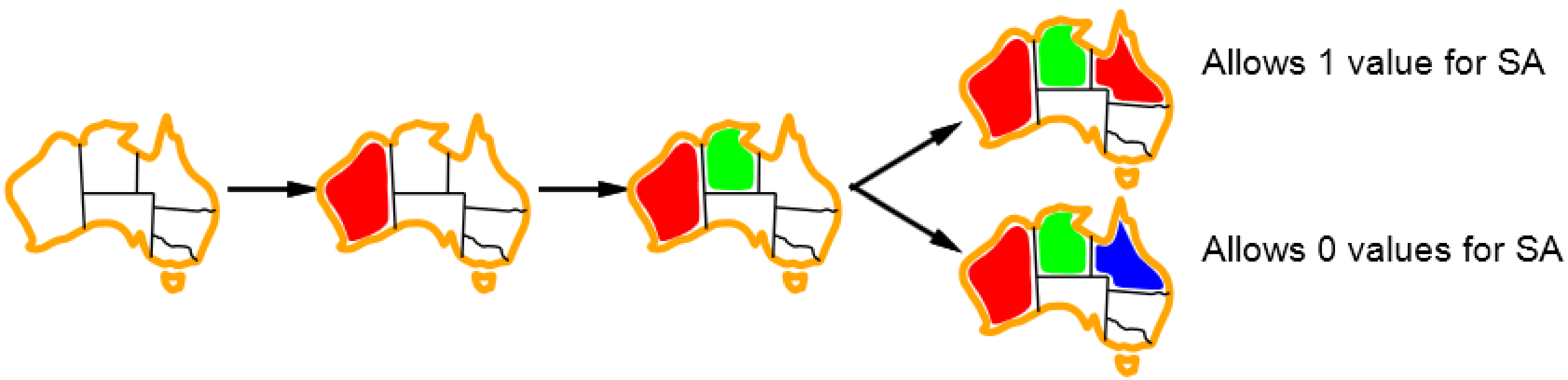

OrderDomainValues

- Least Constraining Value Heuristic
  - Given a variable, choose the value that rules out the fewest values in the remaining variables

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  - Least Constraining Value Heuristic
    - Given a variable, choose the value that rules out the fewest values in the remaining variables
  - used in this order, these three can greatly speed up search
    - e.g., n-queens from 25 queens to 1000 queens

SelectUnassignedVariable

OrderDomainValues

Constraint Propagation - Sudoku

- Problem
  - CSP with 81 variables
- Constraints
  - some values are assigned in the start (unary constraints)
  - 27 constraints on 9 values that must all be different

    (9 rows, 9 columns, 9 squares)
- Constraint Propagation
  - People often write a list of possible values into empty fields
  - try to successively eliminate values
- Status
  - Automated constraint solvers can solve the hardest puzzles in no time

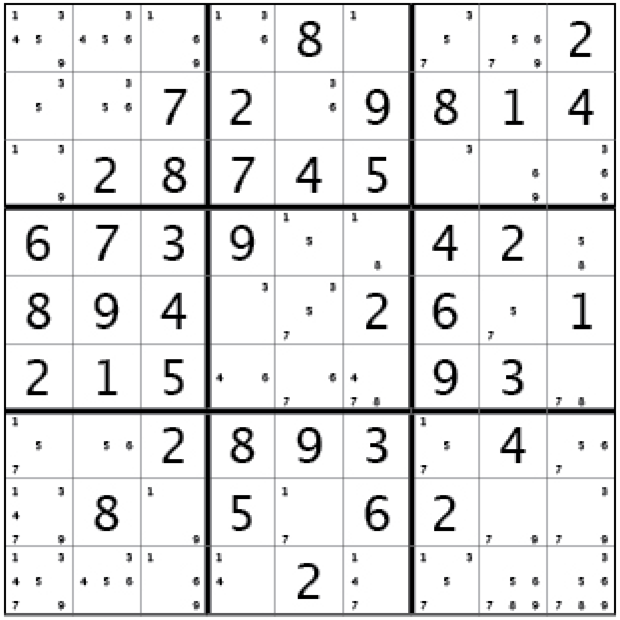


Figure 6

Node Consistency

Node Consistency

- the possible values of a variable must conform to all unary constraints
- can be trivially enforced
- Example:
  - Soduko: Some nodes are constrained to a single value

More General Idea: Local Consistency

- make each node in the graph consistent with its neighbors
- by (iteratively) enforcing the constraints corresponding to the edges

Arc Consistency

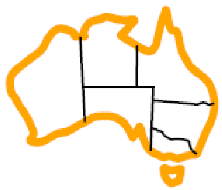
- every domain must be consistent with the neighbors:

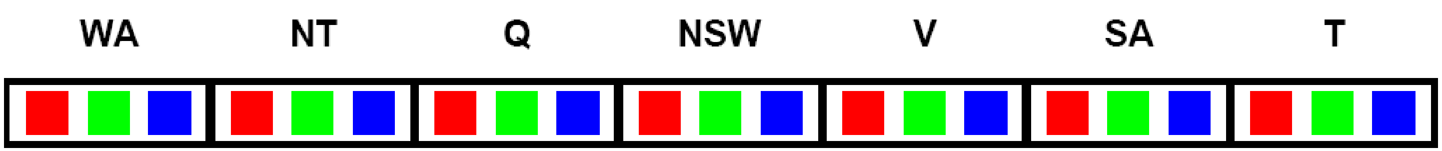
> A variable X_i is arc-consistent with a variable X_j if
> - for every value in its domain D_i
> - there is some value in D_j
> - that satisfies the constraint on the arc (X_i,X_j)

- can be generalized to n-ary constraints
  - each tuple involving the variable X_i has to be consistent

Forward Checking

- Idea: establish arc consistency for every new variable
  - keep track of remaining legal values for unassigned variables
  - terminate search when any variable has no more legal values





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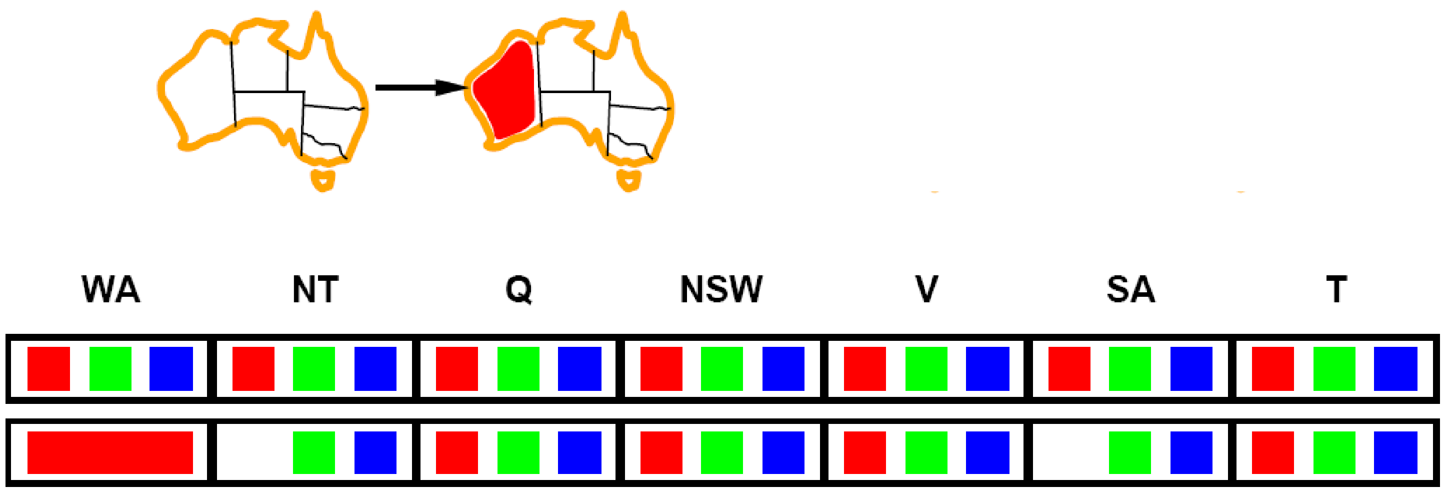

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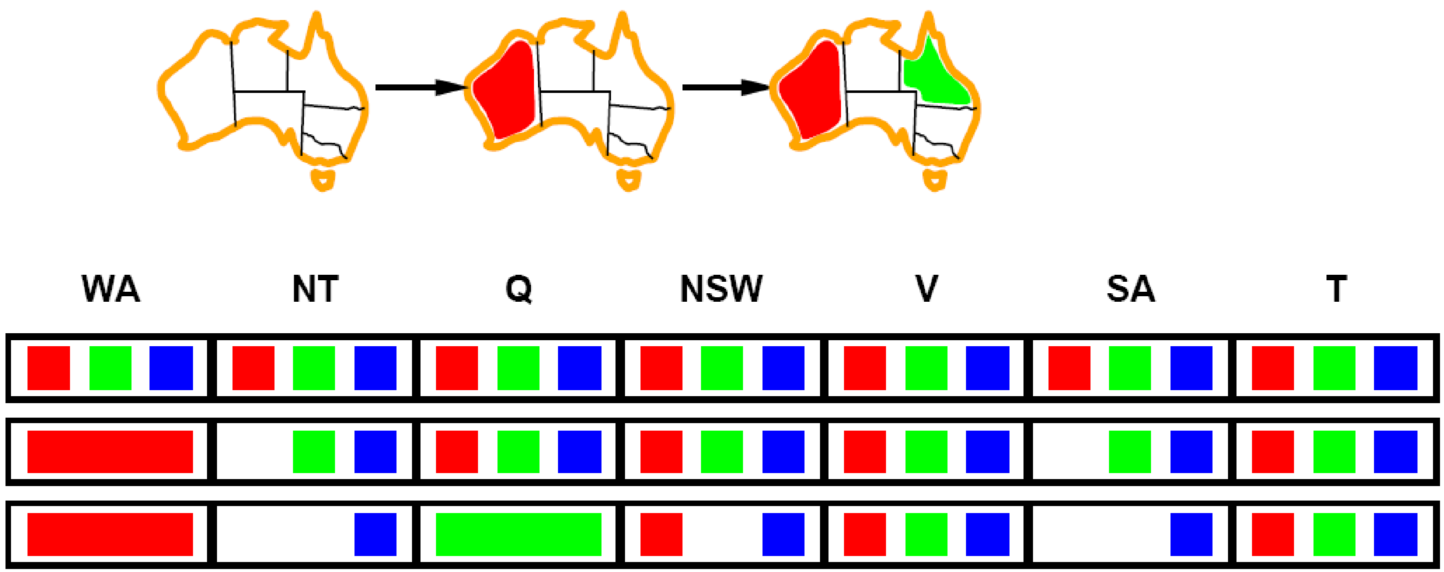

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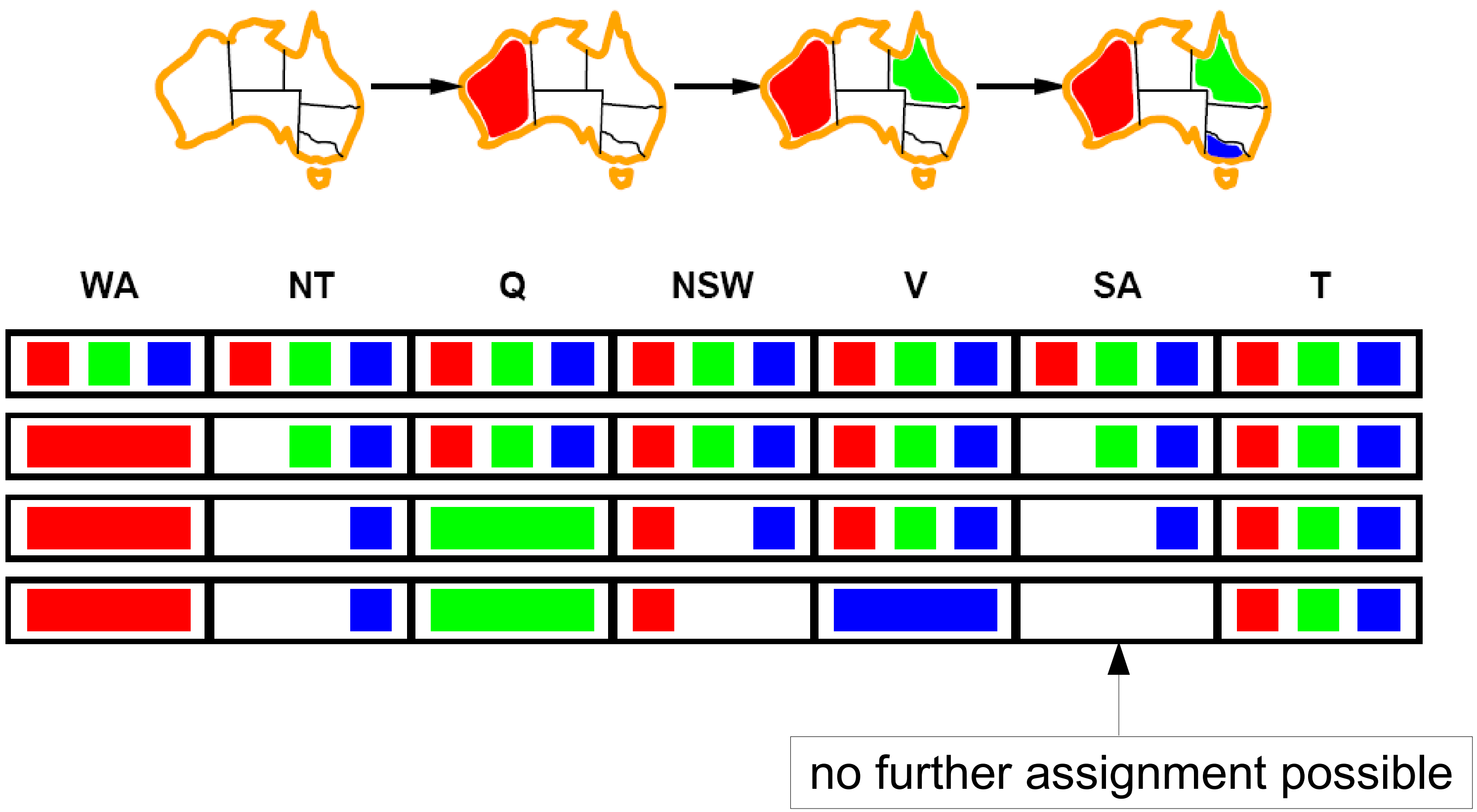
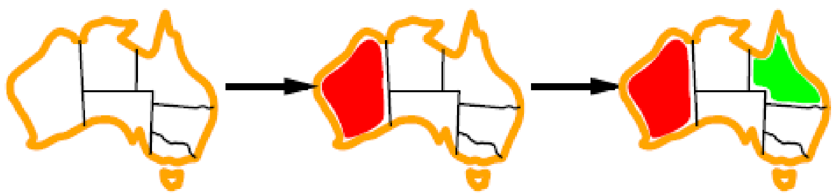
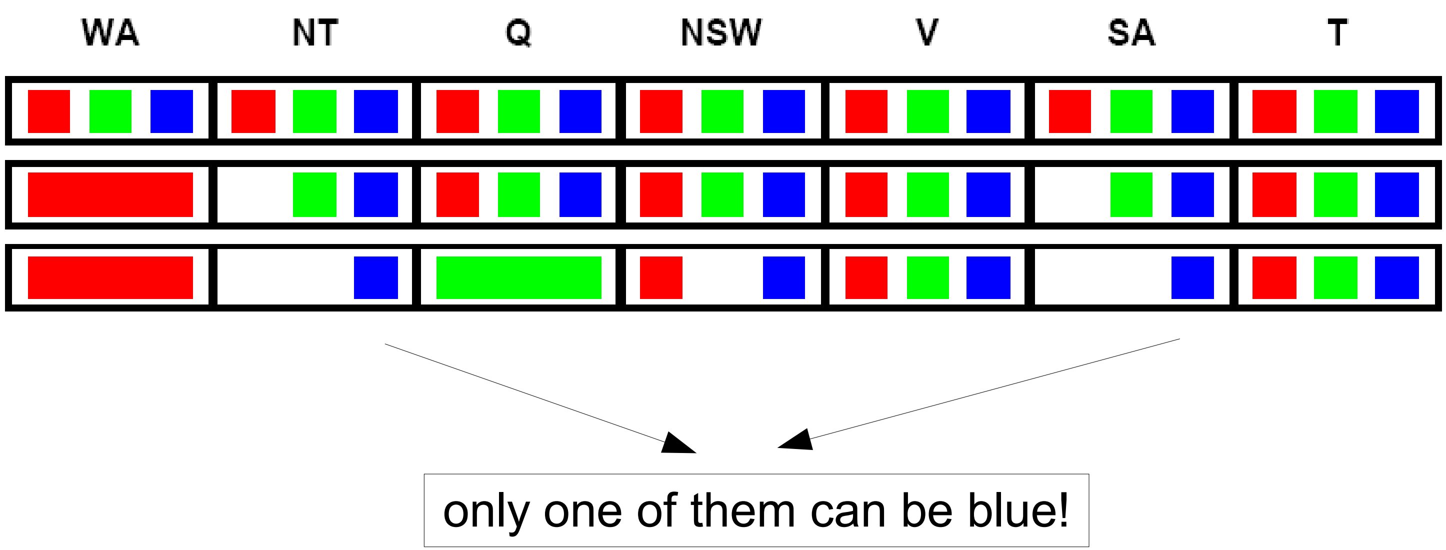


Constraint Propagation

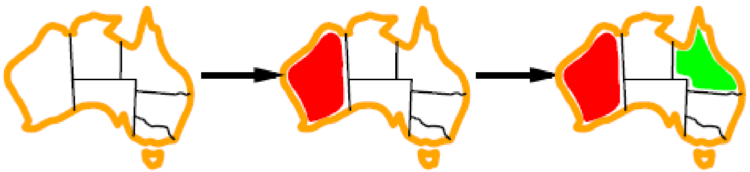
- Problem:
  - forward checking propagates information from assigned to unassigned variables
  - but doesn't look ahead to provide early detection for all failures

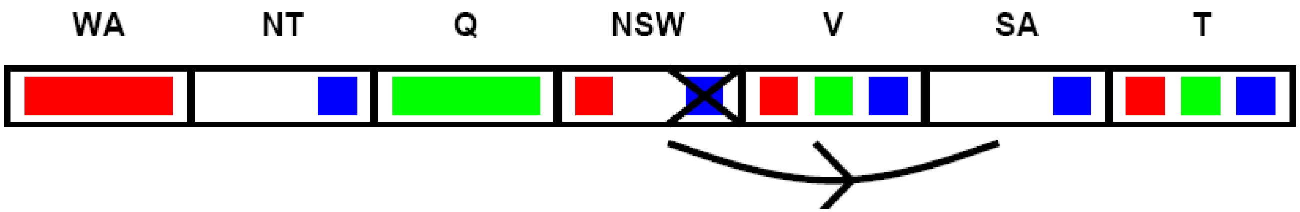




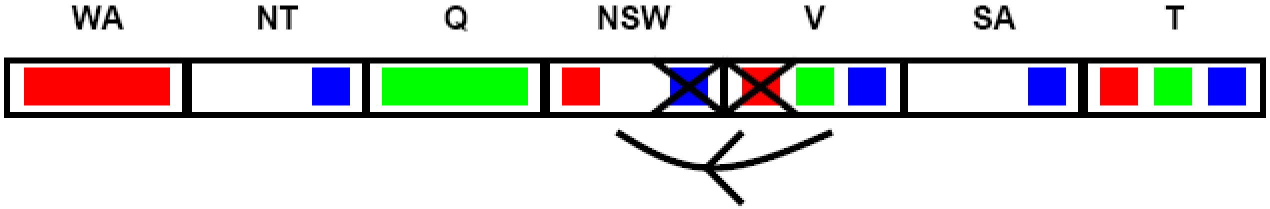
Maintaining Arc Consistency (MAC)

- After each new assignment of a value to a variable, possible values of the neighbors have to be updated:





- If one variable (NSW) looses a value (blue), we need to recheck its neighbors as well:



Arc Consistency Algorithm

```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
```

If X loses a value, neigbors of X need to be rechecked.

```
function REMOVE-INCONSISTENT-VALUES( Xi, Xj) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy the constraint Xi ↔ Xj
         then delete x from DOMAIN[Xi];  removed ← true
   return removed
```

- Run-time: $O(n^2d^3)$ (can be reduced to $O(n^2d^2)$)
  more efficient than forward checking (see later)

Path Consistency

- Arc Consistency is often sufficient to
  - solve the problem (all domains have size 1)
  - show that the problem cannot be solved (some domains empty)
- but may not be enough
  - there is always a consistent value in the neighboring region

→ Path consistency
  - tightens the binary constraints by considering triples of values

> A pair of variables (X_i , X_j) is path-consistent with X_m if
> - for every assignment that satisfies the constraint on the arc (X_i,X_j)
> - there is an assignment that satisfies the constraints on the arcs (X_i,X_m) and (X_j,X_m)

- Algorithm AC-3 can be adapted to this case (known as PC-2)

k-Consistency

- The concept can be generalized so that a set of k values need to be consistent
  - 1-consistency = node consistency
  - 2-consistency = arc consistency
  - 3-consistency = path consistency
  - ....
- May lead to faster solution ($O(n^2d)$)
  - but checking for k-Consistency is exponentional in k in the worst case
- therefore arc consistency is most frequently used in practice

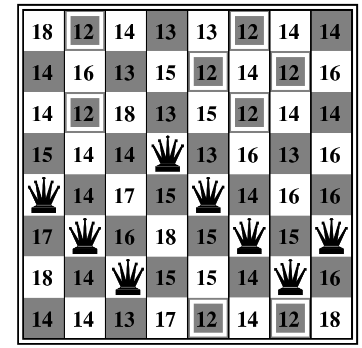
Sudoku

- simple puzzles can be solved with AC-3
  - the 9-valued AllDiff constraints can be converted into pairwise binary constraints (36 each)
  - therefore 27x36 = 972 arc constraints
- somewhat more with PC-2
  - there are 255,960 path constraints
- however, not all problems can be solved with constraint progapagation alone
  - to solve all puzzles we need a bit of search

Integrating Constraint Propagation and Backtracking Search

- Performance of Backtracking can be further sped up by integrating constraint propagation into the search
- Key idea:
  - each time a variable is assigned, a constraint propagation algorithm is run in order to reduce the number of choice points in the search
- Possible algorithms
  - Forward Checking
  - AC-3, but initial queue of constraints only contains constraints with the variable that has been changed

Local Search for CSP



- Modifications for CSPs:
  - work with complete states
  - allow states with unsatisfied constraints
  - operators reassign variable values

Min-conflicts is the heuristic that we studied for the 8-queens problems.

- Min-conflicts Heuristic:
  - randomly select a conflicted variable
  - choose the value that violates the fewest constraints
  - hill-climbing with $h(n)$ = # of violated constraints
- Performance:
  - can solve randomly generated CSPs with a high probability
  - except in a narrow range of

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$